ABSTRACT

The purpose of this chapter is to present the theoretical foundations of multi-moment asset allocation and pricing models in an expected utility framework. Using an infinite-order Taylor series expansion, we first recall the link between the expected utility and higher moments of the investment return distribution (Tsiang, 1972; Loistl, 1976 and Lhabitant, 1997). Following the approach of Benishay (1987, 1989 and 1992) and Rossi and Tibiletti (1996), we next develop a quartic utility specification to obtain an exact mean–variance–skewness–kurtosis decision criterion. We also present the behavioural and distributional conditions under which the preference of a rational agent can be approximated by a fourth-order Taylor series expansion. The Taylor approach and the polynomial utility specification are then compared when justifying a moment-based decision criterion.

1.1 INTRODUCTION

The definition of a decision’s criterion under uncertainty is a prerequisite for the derivation of an equilibrium asset pricing relation. Multi-moment asset allocation and pricing models assume that investors determine their investment by taking into account only the first \( N \) moments of the portfolio return distribution. Agents are supposed to maximise their expected utility\(^1\), which can be represented by an indirect function that is strictly concave and decreasing with even moments and strictly concave and increasing with odd moments.

Despite the tractability and economic appeal of such models, their theoretical justifications are far from simple. First, it is not always possible to translate individual preferences into a function that depends on the entire sequence of the moments of the portfolio return distribution (Loistl, 1976; Lhabitant, 1997 and Jurczenko and Maillet, 2001). Most importantly, agents who maximise their expected utility do not, in general, have preferences that can be translated into a simple comparison of the first \( N \) moments of their investment return.

\(^1\) The expected utility criterion remains the traditional one for rational individual decisions in a risky environment.
distribution. Brockett and Kahane (1992) show\(^2\) that it is always possible to find two random variables such that the probability distribution of the first random variable dominates statistically\(^3\) the second one with respect to the first \(N\) moments, but is stochastically dominated to the \(N\)th order\(^4\) for some rational investors. Since \(N\)th degree stochastic dominance implies a lexicographic order over the first \(N\) moments (Fishburn, 1980; O’Brien, 1984 and Jean and Helms, 1988), the preference ordering will coincide with a moment-based ranking only when all the moments up to order \((N – 1)\) are equal.\(^5\)

Although there is no bijective relation between the expected utility theory and a moment-based decision criterion, it is, however, possible, by suitably restricting the family of distributions and von Neumann–Morgenstern utility functions, to translate individual preferences into a partial moment ordering. Conditional on the assumption that higher moments exist, the expected utility can then be expressed – approximately or exactly – as an increasing function of the mean and the skewness, and a decreasing function of the variance and the kurtosis of the portfolio return distribution.\(^6\)

The purpose of this first chapter is to present the theoretical foundations of a mean–variance–skewness–kurtosis decision criterion. To achieve this, we consider investors endowed with utility functions relevant for the fourth-order stochastic dominance (Levy, 1992 and Vinod, 2004). We first recall the link that exists between the expected value function and the moments of the probability distribution. This leads us to specify the interval of convergence of the Taylor series expansion for most of the utility functions used in the finance field and to characterise the maximum skewness–kurtosis domain for which density functions exist (Hamburger, 1920 and Widder, 1946). We then introduce a quartic parametric utility function to obtain an exact mean–variance–skewness–kurtosis decision criterion (Benishay, 1987, 1989 and 1992). Following the approach of Rossi and Tibiletti (1996) and Jurczenko and Maillet (2001), we show how such polynomial specification can satisfy – over a realistic range of returns – the five desirable properties of utility functions stated by Pratt (1964), Arrow (1970) and Kimball (1990) – non-satiation, strict risk aversion, strict decreasing absolute risk aversion (DARA), strict decreasing absolute prudence (DAP) and constant or increasing relative risk aversion (CRRA or IRRA).

We then present the conditions under which rational preferences can be approximated by a fourth-order Taylor series expansion.

Even though Taylor series approximations or polynomial utility specifications have already been considered to deal with the non-normalities in the asset return distributions (see, for instance, Levy, 1969; Hanoch and Levy, 1970; Rubinstein, 1973 and Kraus and Litzenberger, 1976), this contribution constitutes – to the best of our knowledge – the first one that considers in detail the theoretical justifications of multi-moment asset allocation and pricing models.

The chapter is organised as follows. In Section 1.2 we review the link between the expected utility function and the centred moments of the terminal return distribution. In Section 1.3 we study the preference and distributional restrictions that enable us to express the expected

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3 In the sense that it is characterised – for instance – by a higher expected value, a lower variance, a higher skewness and a lower kurtosis.

4 For a survey on stochastic dominance literature, see Levy (1992).

5 The mean–variance portfolio selection suffers the same flaw, since a mean–variance ordering does not constitute a necessary condition for second-order stochastic dominance (Hanoch and Levy, 1969).

6 Throughout this chapter, we restrict our attention only to the first four moments since there is no clear economic justification concerning the link between the expected utility function and the fifth and higher-order centred moments of the investment return distribution. Moreover, even though most empirical works show that moments up to fourth order exist in (un)conditional asset return distributions (see, for instance, Lux, 2000 and 2001 and Jondeau and Rockinger, 2003a), there remains an issue concerning the existence of moments beyond the fourth. Another problem is the poor sampling properties of higher moments’ empirical estimators due to high powers in the expectation operator (Vinod, 2001 and 2004).
utility criterion as an exact function of the mean, variance, skewness and kurtosis of the portfolio return density. Section 1.4 presents the theoretical justifications of a four-moment expected utility approximation. Section 1.5 concludes. Proofs of all the theorems presented in the chapter are given in the appendices.

### 1.2 Expected Utility and Higher-Order Moments

We consider a one-period single exchange economy with one consumption good serving as numéraire. Each agent has an initial endowment, \( W_0 \), arbitrarily fixed to one without any loss of generality, and a von Neumann–Morgenstern utility function \( U(\cdot) \) defined over its final wealth and denoted \( W_F \), from \( I \subseteq IR \) to \( IR \). The preference function is assumed to belong to the class of utility functions, called \( D_4 \), relevant for the fourth-order stochastic dominance (abbreviated to 4SD)\(^7\), satisfying:

\[
D_4 = \{ \left| U^{(i)}(\cdot) > 0 \right. \} \quad \text{for} \quad i = 1, \ldots, 4
\]

where \( U^{(i)}(\cdot) \) with \( i = [1, \ldots, 4] \) are the derivatives of order \( i \) of \( U(\cdot) \).

At the beginning of the period, each agent maximises the expected utility of its end-of-period investment gross rate of return, denoted \( R \), such that:

\[
E[U(R)] = \int_{-\infty}^{+\infty} U(R) \, dF(R)
\]

where \( F(\cdot) \) is the continuous probability distribution of \( R = W_F / W_0 \).

If the utility function is arbitrarily continuously differentiable in \( I \), one can express the utility of the investor \( U(\cdot) \) as an \( N \)th order Taylor expansion, evaluated at the expected gross rate of return on the investment, that is, \( \forall R \in I \):

\[
U(R) = \sum_{n=0}^{N} (n!)^{-1} U^{(n)}(E(R)) [R - E(R)]^n + \varepsilon_{N+1}(R)
\]

where \( E(R) \) is the expected simple gross rate of return, \( U^{(n)}(\cdot) \) is the \( n \)th derivative of the utility function and \( \varepsilon_{N+1}(\cdot) \) is the Lagrange remainder defined as:

\[
\varepsilon_{N+1}(R) = \frac{U^{(N+1)}(\xi)}{(N+1)!} [R - E(R)]^{(N+1)}
\]

where \( \xi \in ]R, E(R)[ \) if \( R < E(R) \), or \( \xi \in ]E(R), R[ \) if \( R > E(R) \), and \( N \in IN^* \).

---

\(^7\) Let \( X \) and \( Y \) be two continuous random variables defined by their probability distributions \( F_X(\cdot) \) and \( F_Y(\cdot) \). The variable \( X \) is said to dominate stochastically the variable \( Y \) to the fourth-order – that is, \( X \) is preferred over \( Y \) for the class \( D_4 \) of utility functions – if and only if, whatever \( p \):

\[
\begin{align*}
\int_{-\infty}^{0} [F_X(z) - F_Y(z)] \, dz &\leq 0 \\
\int_{-\infty}^{0} \int_{-\infty}^{q} [F_X(z) - F_Y(z)] \, dz \, dq &\leq 0 \\
\int_{-\infty}^{0} \int_{-\infty}^{q} \int_{-\infty}^{r} [F_X(z) - F_Y(z)] \, dz \, dq \, dr &\leq 0
\end{align*}
\]

with \( (p \times q \times r) = (IR)^3 \) and at least one strict inequality over the three for some \( p \).
If we assume, moreover, that the $N$th Taylor approximation of $U(\cdot)$ around $E(R)$ converges absolutely towards $U(\cdot)$, that the integral and summand operators are interchangeable, and that the moments of all orders exist and determine uniquely the return distribution, taking the limit of $N$ towards infinity and the expected value on both sides in (1.3) leads\(^8\) to:

$$
E[U(R)] = \int_{-\infty}^{+\infty} \left\{ \lim_{N \to \infty} \left[ \sum_{n=0}^{N} \frac{(n!)^{-1} U^{(n)}[E(R)] [R - E(R)]^n}{n!} + e_{N+1}(R) \right] \right\} \, dF(R) 
$$

$$
= U[E(R)] + \frac{1}{2} U^{(2)}[E(R)] \sigma^2(R) + \frac{1}{3!} U^{(3)}[E(R)] s^3(R) 
+ \frac{1}{4!} U^{(4)}[E(R)] \kappa^4(R) + \sum_{n=5}^{\infty} \frac{1}{n!} U^{(n)}[E(R)] m^n(R) 
$$

(1.4)

where $\sigma^2(R) = E\left\{ [R - E(R)]^2 \right\}$, $s^3(R) = E\left\{ [R - E(R)]^3 \right\}$, $\kappa^4(R) = E\left\{ [R - E(R)]^4 \right\}$ and $m^n(R) = E\left\{ [(R - E(R))^n] \right\}$ are, respectively, the variance, the skewness, the kurtosis\(^9\) and the $n$th centred higher moment of the investor’s portfolio return distribution, and:

$$
\lim_{N \to \infty} e_{N+1}(R) = 0 
$$

There are three conditions under which it is possible to express a continuously differentiable expected utility function as a function depending on all the moments of the return distribution.

The first condition implies that the utility function $U(\cdot)$ is an analytic function\(^10\) at $E(R)$ and that the realised returns must remain within the absolute convergence interval of the infinite-order Taylor series expansion of the utility function considered (Tsiang, 1972; Loistl, 1976; Hasset et al., 1985 and Lhabitant, 1997).

**Theorem 1**  A sufficient condition for a Taylor series expansion of an infinitely often differentiable utility function $U(\cdot)$ around the expected gross rate of return $E(R)$ to converge absolutely is that the set of realisations of the random variable $R$ belongs to the open interval $J$ defined by:

$$
|R - E(R)| < \zeta 
$$

(1.5)

---

\(^8\) Under the same set of conditions, it is also possible to express the generic expected utility as a function of the non-centred moments of the return distribution through a MacLaurin series expansion (see, for instance, Levy and Markowitz, 1979; Rossi and Tibiiletti, 1996 and Lhabitant, 1997).

\(^9\) These definitions of skewness and kurtosis, as third- and fourth-order centred moments, differ from the statistical ones as standardised centred higher moments, that is:

$$
\alpha_n = E\left\{ \left[ \frac{R - E(R)}{\sigma(R)} \right]^n \right\} 
$$

with $n = [3,4]$.

\(^10\) A real function $f(x)$ is analytic at $x = a$ if there exists a positive number $\zeta$ such that $f(\cdot)$ can be represented by a Taylor series expansion in the interval $[-\zeta, \zeta]$, centred around $a$, that is, if $\forall x \in [-\zeta, \zeta]$, we have:

$$
f(x) = \sum_{n=0}^{N} \frac{(n!)^{-1} f^{(n)}(a) \times (x - a)^n}{n!} 
$$

where $\zeta$ is called the radius of convergence of the Taylor series of $f(\cdot)$ around $a$, and $f^{(n)}(\cdot)$ is the $n$th derivative of the function $f(\cdot)$. 

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with:

$$\zeta = \lim_{N \to \infty} \frac{(N + 1)! U^{(N)}[E(R)]}{N! U^{(N+1)}[E(R)]}$$

where \( \zeta \) is a positive constant corresponding to the radius of convergence of the Taylor series expansion of \( U(.) \) around \( E(R) \) and \( N \in \mathbb{IN} \).

**Proof**  see Appendix A

The region of absolute convergence of a Taylor series expansion depends on the utility function considered. For instance, condition (1.5) does not require any specific restriction on the return range for exponential and polynomial utility functions since their convergence intervals are defined on the real line, whilst it entails the following restriction for logarithm-power type utility functions (see Table 1.1):

$$0 < R < 2E(R)$$

Even though this condition is not binding for traditional asset classes when short-sale is forbidden, it may be too restrictive for some applications on alternative asset classes due to their option-like features and the presence of leverage effects (Weisman, 2002 and Agarwal and Naik, 2004). In this case, *ex post* returns might lie outside of the convergence interval (1.5).

The second condition entails shrinking the interval of absolute convergence (1.5) slightly so that the infinite-order Taylor series expansion of \( U(.) \) around \( E(R) \) converges uniformly towards \( U(.) \) and the integral and summation operators are interchangeable in the investor’s objective function$^{11}$ (Loistl, 1976; Lhabitant, 1997 and Christensen and Christensen, 2004).

**Table 1.1** Taylor series expansion absolute convergence interval

<table>
<thead>
<tr>
<th>Utility function ( U(R) )</th>
<th>Radius of absolute convergence ( \zeta )</th>
<th>Convergence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>HARA ( \frac{\gamma(b+\frac{x R}{a})^{(1-\gamma)}}{(1-\gamma)} )</td>
<td>(</td>
<td>-\gamma</td>
</tr>
<tr>
<td>CARA (-\exp(-aR))</td>
<td>( +\infty )</td>
<td>( IR )</td>
</tr>
<tr>
<td>CRRA ( \frac{E^{(1-\gamma)}}{(1-\gamma)} )</td>
<td>( E(R) )</td>
<td>( ]0, 2E(R)[ )</td>
</tr>
<tr>
<td>CRRA ( \ln(R) )</td>
<td>( E(R) )</td>
<td>( ]0, 2E(R)[ )</td>
</tr>
<tr>
<td>Polynomial of order ( N ) ( \left(\frac{1-x}{N}\right)\left(b + \frac{aR}{1-x}\right)^{N} )</td>
<td>( +\infty )</td>
<td>( IR )</td>
</tr>
</tbody>
</table>

$^{11}$ Uniform convergence is a sufficient condition for a term-by-term integration of an infinite series of a continuous function.
Theorem 2  A sufficient condition for a Taylor series expansion of an infinitely often differentiable utility function $U(.)$ around the expected gross rate of return $E(R)$ with a positive radius of convergence $\zeta$ to converge uniformly is that the set of realisations of the random variable $R$ remains in the closed interval $J^*$ defined by (using previous notation):

$$|R - E(R)| \leq \zeta^* \quad (1.6)$$

$\forall \zeta^* \in ]0, \zeta[,$ where $\zeta$ is defined in (1.5).

Proof  see Appendix B.

The third condition is related to the Hamburger moment problem (1920), that is, to the question of the existence and uniqueness, for a sequence of non-centred moment constraints $(\mu^k)$, of an absolutely continuous positive distribution probability function $F(.)$ such that

$$\forall k \in {\mathbb N}^* : \mu^k = \int_{-\infty}^{\infty} R^k dF(R) \quad (1.7)$$

where $p := -\infty$, $q := +\infty$ and $\mu^0 = 1$.

If $F(.)$ is unique, the Hamburger moment problem is said to be “determinate”, since the probability distribution function is uniquely determined by the sequence of its moments.

Theorem 3  (Hamburger, 1920; Widder, 1946 and Spanos, 1999). The sufficient conditions for a sequence of non-centred moments $(\mu^k)$, with $0 \leq k \leq 2N$, to lead to a unique continuous positive probability distribution are that:

$$\begin{cases} E\left[|R - E(R)|^k\right] < \infty & \text{(existence of the moments of order } k) \\ \det(\mu^n) \geq 0 \ \forall n \in \{0, \ldots, N\} & \text{(existence of the density function)} \\ - (1 + R)^{-2} \int_{-\infty}^{+\infty} \ln f(R) \, dR = \infty & \text{(Krein's uniqueness condition)} \end{cases} \quad (1.8)$$

where $\mu^n$ is the $[(n + 1) \times (n + 1)]$ Hankel matrix of the non-centred moments defined as:

$$\mu^n = \begin{pmatrix} \mu^0 & \mu^1 & \cdots & \mu^n \\ \mu^1 & \mu^2 & \cdots & \mu^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu^n & \mu^{n+1} & \cdots & \mu^{2n} \end{pmatrix}$$

with elements $(\mu^{i+j})$ with coordinates $(i, j)$, where $(i, j) = [0, \ldots, n]^2$, $\mu^0 = 1; N \in {\mathbb N}^*$ and $f(.)$ is the continuous density function of the gross rate of return $R$.


---

12 The moment problem (1.7) in the case where $p := 0$ and $q := +\infty$ is called the Stieltjes moment problem (Stieltjes, 1894), while in the case where $p := 0$ and $q := 1$, we talk about the little Hausdorff moment problem (Hausdorff, 1921a and 1921b).

13 For necessary and sufficient conditions for uniqueness in the Hamburger moment problem, see Lin (1997) and Stoyanov (1997 and 2000).

14 This approach is based on functional analysis. Another approach to the moment problem involves Padé approximants (see, for instance, Baker and Graves-Morris, 1996).
Conditional on the existence of the moments, the existence of a density function means, for \( k = 4 \), that the sequence of determinants of the associated Hankel matrices \( (\mu^{i+j}) \), with \((i \times j) = [1, 2]^2\), must satisfy:

\[
\det \left( \begin{array}{cc} \mu^0 & \mu^1 \\ \mu^1 & \mu^2 \end{array} \right) \geq 0 \quad \text{and} \quad \det \left( \begin{array}{cccc} \mu^0 & \mu^1 & \mu^2 & \mu^3 \\ \mu^1 & \mu^2 & \mu^3 & \mu^4 \\ \mu^2 & \mu^3 & \mu^4 & \mu^5 \\ \mu^3 & \mu^4 & \mu^5 & \mu^6 \end{array} \right) \geq 0
\]

(1.9)

with \( \mu^0 = 1 \).

Using the following results (Kendall, 1977, p. 58):

\[
\begin{align*}
\mu^2 &= \sigma^2 (R) + [E (R)]^2 \\
\mu^3 &= s^3 (R) + 3E (R) \sigma^2 (R) + [E (R)]^3 \\
\mu^4 &= \kappa^4 (R) + 4E (R) s^3 (R) + 6 [E (R)]^2 \sigma^2 (R) + [E (R)]^4
\end{align*}
\]

(1.10)

the positive definiteness of (1.9) implies the following restriction for the four-moment problem:

\[
(\gamma_1)^2 \leq \gamma_2 + 2
\]

(1.11)

where \( \gamma_1 \) and \( \gamma_2 \) are the Fisher parameters for the skewness and the kurtosis:

\[
\gamma_1 = \frac{s^3 (R)}{\sigma^3 (R)} \quad \text{and} \quad \gamma_2 = \frac{\kappa^4 (R)}{\sigma^4 (R)} - 3
\]

This relation confirms that, for a given level of standardised kurtosis, only a finite range of standardised skewness may be spanned. That is, for \( \gamma_2 \geq (-2) \), the possible standardised skewness belongs to \([ -\gamma_1^*, \gamma_1^* ]\), where:

\[
\gamma_1^* = \sqrt{\gamma_2 + 2}
\]

(1.12)

Figure 1.1 represents the skewness–kurtosis domain ensuring the existence of a density function\(^{15}\) compared to some empirical couples of Fisher coefficient estimates. The curve corresponds to the theoretical maximum domain of the Fisher parameters \( \gamma_1 \) and \( \gamma_2 \), for which density functions exist when the first four moments are given.

While the domain of existence of a density (1.12) is large, the solution of the Hamburger moment problem, if it exists, may not be unique. The Krein’s integral test (third line in equation (1.8)) is not fulfilled by all probability distributions. For instance, the lognormal distribution has finite centred moments of all orders and verifies the inequality (1.11) but is not uniquely determined by its moments. It is indeed straightforward to show that the probability distribution whose density function is given by:

\[
f (R) = (2 \pi)^{-1/2} R^{-1} \exp \left[ -\frac{1}{2} (\ln R)^2 \right] \times [1 + a \sin (2 \pi \ln R)]
\]

(1.13)

\(^{15}\) In the case of a standardised random variable with a zero mean and a unit variance, condition (1.11) leads to (see Jondeau and Rockinger, 2003a):

\[
(\mu^3)^2 \leq \mu^4 - 1
\]

where \( \mu^0 = 1 \) and \( \mu^1 = 0 \).
Figure 1.1  Realistic Fisher coefficients domain for return density functions. This figure illustrates the boundaries of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ Fisher coefficients ensuring the existence of a density function when the first four moments are given (see Equation 1.12). The black (white) region in the figure represents (60 000) simulated couples of Fisher ($\hat{\gamma}_1$, $\hat{\gamma}_2$) parameters that violate (or respect) the condition of existence of a density function in the four-moment case. The grey lozenges correspond to the (2432) simple weekly rolling (overlapping) estimates of the two first Fisher coefficients on the daily returns of random constrained (positive and unitary sum) portfolios composed with close-to-close CAC40 stocks on the period 01/95 to 08/04. See Jondeau and Rockinger (2003a, Figure 5, p. 1707) for a comparison with normalised variables and generalised $t$-student skewness and kurtosis boundaries.

for $|a| < 1$, has exactly the same sequence of moments as the lognormal distribution obtained by substituting $a = 0$ in (1.13) – see Heyde (1963).

Under these regular conditions, any investor with a utility function belonging to the $D_4$ class displays a preference for the expected return and the (positive) skewness, and an aversion to the variance and the kurtosis, since differentiating equation (1.4) with respect to $E(R)$, $\sigma^2(R)$, $m^3(R)$ and $\kappa^4(R)$, leads to, $\forall R \in [E(R) - \xi^*, E(R) + \xi^*]$:

\[
\begin{align*}
\frac{\partial E[U(R)]}{\partial E(R)} &= (n!)^{-1} \sum_{n=0}^{\infty} U^{(n+1)}[E(R)] m^n(R) = E[U^{(1)}(R)] \quad (> 0) \\
\frac{\partial E[U(R)]}{\partial \sigma^2(R)} &= (2!)^{-1} U^{(2)}[E(R)] \quad (< 0) \\
\frac{\partial E[U(R)]}{\partial m^3(R)} &= (3!)^{-1} U^{(3)}[E(R)] \quad (> 0) \\
\frac{\partial E[U(R)]}{\partial \kappa^4(R)} &= (4!)^{-1} U^{(4)}[E(R)] \quad (< 0)
\end{align*}
\]

(1.14)

where $\xi^*$ is defined as in (1.6).
Standard risk aversion (see Kimball, 1993) and strict consistency in the direction of preference for higher moments (see Horvath and Scott, 1980) constitute two sufficient, but non-necessary, conditions to establish a mean–variance–skewness–kurtosis preference ordering for a non-satiable and risk-averse investor.

Indeed, Kimball (1993) shows that decreasing absolute risk aversion (DARA) and decreasing absolute prudence (DAP) are sufficient conditions for any monotonically increasing and strictly concave utility function to belong to the standard risk-aversion utility class. This implies the mean–variance–skewness–kurtosis preference relation (1.14), since, \( \forall R \in J^* \):

\[
\begin{align*}
\frac{d}{dR} \left( -\frac{U^{(2)}(R)}{U^{(1)}(R)} \right) &= \frac{-U^{(3)}(R) U^{(1)}(R) + \left[ U^{(2)}(R) \right]^2}{\left[ U^{(1)}(R) \right]^2} < 0 \implies U^{(3)}(R) > 0 \\
\frac{d}{dR} \left( -\frac{U^{(3)}(R)}{U^{(2)}(R)} \right) &= \frac{-U^{(4)}(R) U^{(2)}(R) + \left[ U^{(3)}(R) \right]^2}{\left[ U^{(2)}(R) \right]^2} < 0 \implies U^{(4)}(R) < 0
\end{align*}
\]

with \( U^{(1)}(.) > 0 \) and \( U^{(2)}(.) < 0 \).

Horvath and Scott (1980) reach a similar result under the hypotheses of non-satiation, risk aversion and strict consistency in preference direction for higher moments. Strict consistency of an investor’s preference with respect to higher moments requires that all the \( n \)th derivatives of \( U(.) \) are either always negative, always positive or everywhere zero for any \( n \geq 3 \) and every possible return. That is, \( \forall n \geq 3 \) and \( \forall R \in J^* \):

\[
\begin{align*}
U^{(n)}(R) < 0 \\
U^{(n)}(R) = 0 \quad \text{or} \\
U^{(n)}(R) > 0
\end{align*}
\]

where \( J^* \) is the interval of uniform convergence of the Taylor series expansion of \( U(.) \) around \( E(R) \) defined in (1.6). Under non-satiation and risk aversion, the next preference restrictions then follow from the mean-value theorem (see Horvath and Scott, 1980, p. 916):

\[
(-1)^n U^{(n)}(R) < 0
\]

\( \forall n \geq 3 \) and \( \forall R \in J^* \). In particular, if \( U^{(3)}(R) < 0 \ \forall R \in J^* \), the assumption of positive marginal utility is violated for all feasible gross rates of return; so we must have \( U^{(3)}(R) > 0 \ \forall R \in J^* \). Likewise, \( U^{(4)}(R) > 0 \ \forall R \in J^* \), would violate the assumption of strict risk aversion.

More generally, non-satiation, risk aversion and strict consistency of agent preferences towards higher moments imply a preference for expected return and (positive) skewness and an aversion for variance and kurtosis, and more generally a preference for odd higher-order centred moments and an aversion to even higher-order centred moments.

Provided that the utility functions of agents belong to the class of the analytic utility functions relevant for the fourth-order stochastic dominance, that the realisations of the investment returns remain in the interval of the uniform convergence of the Taylor series expansions of the utility functions, and that the Hamburger moment problem is
“determinate”, it is then possible to express the expected utility criterion as a function that depends positively on the mean, skewness and odd higher-order moments and negatively on the variance, kurtosis and even higher-order moments of investment rates of return.

1.3 EXPECTED UTILITY AS AN EXACT FUNCTION OF THE FIRST FOUR MOMENTS

While, under some behavioural and distributional restrictions, individual preferences are linked with the first four moments of the investment return distribution, the expected utility of such investors also depends upon all the other moments of their portfolio returns, so that the mean–variance–skewness–kurtosis approach is necessarily restrictive.16 As in the mean–variance case, there are two ways one can theoretically justify an exact four-moment decision criterion in an expected utility framework: the first theoretical justification consists in restricting asset return distributions to a specific four-parameter distribution class, while the second one consists in restricting investors’ utility functions to a quartic polynomial specification.

The first theoretical justification of a mean–variance–skewness–kurtosis decision criterion is to assume that the asset return distributions belong to a four-parameter family of probability distributions that allows for finite mean, variance, skewness and kurtosis. Without pretence of exhaustivity, we can underline the choice of skew-student distributions (Azzalini and Capitanio, 2003; Branco et al., 2003; Adcock, 2003 and Harvey et al., 2004), normal inverse Gaussian (Barndorff-Nielsen, 1978 and 1997 and Eriksson et al., 2004), confluent U hypergeometric distributions (Gordy, 1998), generalised beta distributions of the second kind (Bookstaber and McDonald, 1987; McDonald and Xu, 1995 and Dutta and Babbel, 2005), Pearson type IV distributions (Bera and Premaratne, 2001), four-moment maximum entropy distributions (Jondeau and Rockinger, 2002; Bera and Park, 2003 and Wu, 2003), Gram–Charlier and Edgeworth statistical series expansions (Jarrow and Rudd, 1982; Corrado and Su, 1996a and 1996b; Capelle-Blanchard et al., 2001 and Jurczenko et al., 2002a, 2002b and 2004), copula-based multivariate non-elliptical distributions (Patton, 2001 and 2004; Xu, 2004) and multivariate separable skewed fat-tailed distributions (Ortobelli, 2001 and Ortobelli et al., 2000 and 2002).

But, if the normal density is perfectly and uniquely defined by its first two moments, this is not the case when some other density functions are considered. This last point leads, in fact, to real difficulties. For instance, as shown by Brockett and Kahane (1992), Simaan (1993) and Brockett and Garven (1998), it is unlikely to determine, on a priori grounds, the sign of sensitivities of the objective function with respect to the moments of the portfolio return distribution. Indeed, when higher-order moments are not orthogonal one to another, the effect on the utility function of increasing one of them becomes ambiguous. Moreover, even if the considered density function can be in accordance with some of the stylised facts highlighted in the literature17, additional utility assessments

16 The mean–variance criterion suffers the same flaw (see Markowitz, 1952).
17 Such as leptokurticity, asymmetry, time-aggregation properties of the process, the leverage effect and clustering in the volatility (Cont, 2001).
are required to obtain an ordering consistent with the fourth-order stochastic dominance criterion.

A second theoretical justification of the mean–variance–skewness–kurtosis analysis is to consider a fourth-order polynomial utility specification. In this case, conditional on the assumption that kurtosis exists and provided that the range of returns is well-restricted, the expected utility ordering can be translated exactly into a four-moment ordering.

A quartic parametric utility function can be defined as follows\(^{18}\) (see Benishay, 1987, 1989 and 1992):

\[
U(R) = a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4
\]  

(1.18)

with \(a_i \in IR^*, \, i = [1, \ldots, 4]\).

Since all partial derivatives above the fourth order are equal to zero, taking the expected value of (1.18) yields:

\[
E[U(R)] = a_0 + a_1 E(R) + a_2 E(R^2) + a_3 E(R^3) + a_4 E(R^4)
\]  

(1.19)

Using the results about uncentred moment definitions recalled from (1.10), we obtain:

\[
E[U(R)] = a_0 + a_1 E(R) + a_2 E(R^2) + a_3 E(R^3) + a_4 E(R^4)
\]  

+ \[a_2 + 3a_3 E(R) + 6a_4 E(R)^2\] \(\sigma^2(R)\)

+ \[a_3 + 4a_4 E(R)\] \(s^3(R)\) + \(a_4 \kappa^4(R)\)

(1.20)

When the first four moments exist and determine uniquely the return distributions (see above), the investors’ preferences can then be expressed as an exact function of the mean, variance, skewness and kurtosis of their portfolio returns. Taking the first four derivatives of the quartic utility function and using the behavioural requirements associated with the fourth-order stochastic dominance criterion defined in (1.1) leads to the following theorem.

**Theorem 4** The necessary and sufficient conditions for a quartic utility function \(U(.)\) to belong to the class \(D_4\) of utility functions relevant for the fourth-order stochastic dominance are given by the following system:

\[^{18}\text{Since a von Neumann–Morgenstern utility function is uniquely defined up to an increasing affine transformation, it is always possible to give a simpler and equivalent expression for the quartic utility function. Substracting }a_0\text{ to equation (1.18) and dividing it by }a_1, \text{ we have:}\]

\[U(R) = R + bR^2 + cR^3 + dR^4\]

where \(b = a_2/a_1, c = a_3/a_1, d = a_4/a_1\) and \(a_1 > 0\).
\[
\begin{align*}
&\begin{cases}
a_1 > 0 \\
\left(-\frac{a_3^2}{16a_4^2} + \frac{a_2}{6a_4}\right)^3 + \left(\frac{a_3^2}{16a_4^2} + \frac{a_1}{8a_4} - \frac{a_2a_3}{16a_4^2}\right)^2 > 0 \quad \text{(non-satiable individuals)} \end{cases} \\
R < -\frac{a_3}{4a_4} + \frac{K + \left(9a_3^2 - 24a_4a_2\right)K^{-1}}{12a_4} \\
0 > a_2 \geq \left(\frac{3a_3^2}{8a_4}\right) \\
R < -\frac{a_3}{4a_4} + \frac{\sqrt{9a_3^2 - 24a_4a_2}}{12a_4} \quad \text{and} \quad R > -\frac{a_3}{4a_4} - \frac{\sqrt{9a_3^2 - 24a_4a_2}}{12a_4} \\
or a_2 < \left(\frac{3a_3^2}{8a_4}\right) \\
a_3 > 0 \\
R < -\left(\frac{a_3}{4a_4}\right) \quad \text{(prudent investors)} \\
a_4 < 0 \\
\end{align*}
\]

with:
\[
K = \left(\frac{A + \sqrt{-108\left(3a_3^2 - 8a_4a_2\right)^3 + A^2}}{2}\right)^{1/3}
\]

and:
\[
A = -54a_3^3 - 432a_4^2a_1 + 216a_4a_3a_2
\]

where \(a_i \in \mathbb{R}^+, i = [1, \ldots, 4]\).

**Proof** see Appendix C.

Any investor with a \(D_4\) class quartic utility function has a preference for the mean, an aversion to the variance, a preference for the (positive) skewness and an aversion to the kurtosis, that is:

\[
\begin{align*}
\frac{\partial E[U(R)]}{\partial E(R)} &= a_1 + 2a_2E(R) + 3a_3E(R^2) + 4a_4E(R^3) > 0 \\
\frac{\partial E[U(R)]}{\partial \sigma^2(R)} &= a_2 + 3a_3E(R) + 6a_4\left[ E(R^2) + \sigma^2(R) \right] < 0 \\
\frac{\partial E[U(R)]}{\partial m^3(R)} &= a_3 + a_4E(R) > 0 \\
\frac{\partial E[U(R)]}{\partial \kappa^4(R)} &= a_4 < 0
\end{align*}
\]
The regularity conditions in (1.22) lead, nevertheless, to some restrictions on the asset return realisations. Indeed, when we restrict investors’ preferences to a quartic specification, we have to make sure that the asset return realisations belong to the interval where the utility function exhibits non-satiety, strict risk aversion, strict prudence and strict temperance. In other words, the *ex post* investor’s portfolio gross rate of return must respect the following system of inequalities:

\[
R < -\frac{a_3}{4a_4} + \min \left[ \Delta, \frac{K + (9a_3^2 - 24a_4a_2) K^{-1}}{12a_4} \right]
\] (1.23)

with:

\[
\Delta = \min \left( \frac{\sqrt{9a_3^2 - 24a_4a_2}}{12a_4}, 0 \right)
\]

and:

\[
\left\{ \left( -\frac{a_3^2}{16a_4^2} + \frac{a_2}{6a_4} \right)^3 + \left( \frac{a_3^2}{16a_4^2} - \frac{a_2a_3}{16a_4^2} + \frac{a_1}{8a_4} \right)^2 \right\} > 0
\]

where \( a_i \in IR^*, i = [1, \ldots, 4] \).

Figure 1.2 represents three particular quartic utility functions and the evolution of their first derivatives with respect to the gross rate of return on a realistic range of returns.

These restrictions constitute necessary, but not sufficient, conditions for the quartic utility function (1.18) to satisfy simultaneously the properties of decreasing absolute risk aversion (DARA), decreasing absolute prudence (DAP) and constant or increasing relative risk aversion (CRRA or IRRA) with respect to the gross rate of return, \( R \). The necessary and sufficient conditions are that the expressions \( \Delta^*, \Delta^{**} \) and \( \Delta^{***} \) respect the following system:

\[
\begin{align*}
\Delta^* &= \left[ -\frac{\partial U(R)}{\partial R} \frac{\partial^3 U(R)}{\partial R^3} + \left( \frac{\partial^2 U(R)}{\partial R^2} \right)^2 \right] < 0 \quad \text{(DARA)} \\
\Delta^{**} &= \left[ -\frac{\partial^2 U(R)}{\partial R^2} \frac{\partial^4 U(R)}{\partial R^4} + \left( \frac{\partial^3 U(R)}{\partial R^3} \right)^2 \right] < 0 \quad \text{(DAP)} \\
\Delta^{***} &= \left[ R\Delta^* - \frac{\partial^2 U(R)}{\partial R^2} \left( \frac{\partial U(R)}{\partial R} \right)^{-1} \right] \geq 0 \quad \text{(CRRA or IRRA)}
\end{align*}
\] (1.24)

This system of inequalities leads to the following theorem.

**Theorem 5** The necessary and sufficient conditions for a quartic utility function \( U(.) \) to exhibit a decreasing absolute risk aversion (DARA), a decreasing absolute prudence (DAP)
First Derivative of a Quartic Utility Function
(Satisty Characteristic)
\[ U^{(1)}(R) = a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4 \]
∀ \( R \in [40\%, 160\%] \), \( U^{(1)}(R) > 0 \), for \( k = 1, \ldots, 3 \)

Second Derivative of a Quartic Utility Function
(Risk-aversion Characteristic)
\[ U^{(2)}(R) = 2a_2 R + 6a_3 R^2 + 12a_4 R^3 \]
∀ \( R \in [40\%, 160\%] \), \( U^{(2)}(R) > 0 \), \( U^{(2)}(R) < 0 \)

Figure 1.2  Quartic utility function illustrations. (a) displays examples of realistic quartic utility functions: \( U_k(R) = a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4 \) for the cases of: 1. \( k = 1 \) non-satiable, risk seeker, prudent and temperant investor; 2. \( k = 2 \) non-satiable, risk averse, imprudent and temperant investor; 3. \( k = 3 \) non-satiable, risk averse, prudent and temperant investor; with:

\[
\begin{align*}
U_1(R) : a_{01} &= 0.3774; a_{11} = 0.0250; a_{21} = 0.0605; a_{31} = 0.0450; a_{41} = 0.0002 \\
U_2(R) : a_{02} &= 0.1954; a_{12} = 0.5230; a_{22} = -0.0013; a_{32} = -0.0695; a_{42} = 0.0019 \\
U_3(R) : a_{03} &= 0.0714; a_{13} = 1.0165; a_{23} = -0.5400; a_{33} = 0.1400; a_{43} = -0.0200
\end{align*}
\]

where coefficient \( a_{jk} \) values, \( j = [0, \ldots, 4] \), result from a grid search on the possible domain, before being rescaled in order to get, for the three utility functions, the same minimum (0.4) and maximum (0.758) score values when the gross return stands respectively at 40 % and 160 % (corresponding to the maximum drawdown and maximum run-up observed on the CAC40 index for six-month series on the period 01/95 to 06/04). See, for comparison, real-market GMM estimates of \( a_{3k} \) and \( a_{4k} \) (with \( a_{0k} = 0, a_{1k} = 1 \) and \( a_{4k} \) unconstrained) in the case of a cubic utility function, in Levy et al. (2003), Table 2 and Figure 1, pp. 11 and 13. The four graphs in (b), (c), (d) and (e) show the evolution of the first derivatives of the three considered quartic utility functions with respect to the gross rate of return on a realistic range of gross returns.
and a constant or increasing relative risk aversion (CRRA or IRRA) with respect to the gross rate of return \( R \) are given by:

\[
\begin{align*}
8a_2^2R^4 + 8a_4a_3R^3 + 3a_2^2R^2 + 2(a_3a_2 - 2a_4a_1)R + \frac{2}{3}a_2^3 - a_3a_4 &< 0 \quad \text{(DARA)} \\
\frac{a_3}{4a_4} + \frac{\sqrt{-9a_3^2 + 24a_4a_2}}{12a_4} &< R < -\frac{a_3}{4a_4} - \frac{\sqrt{-9a_3^2 + 24a_4a_2}}{12a_4} \quad \text{(DAP)} \\
120(a_4)^2R^4 + 120a_3a_4R^3 + 3 & \left[ 9(a_3)^2 + 16a_2a_4 \right] R^2 \\
& + 6(3a_2a_3 + 2a_4a_1)R + 3a_3a_1 + 2(a_2)^2 & \geq \\
& \left( 12a_4R^2 + 6a_3R + 2a_2 \right) \left( 4a_4R^4 + 3a_3R^3 + 2a_2R^2 + a_1R \right)^{-1} \quad \text{(CRRA or IRRA)}
\end{align*}
\]

where \( a_i \in IR^*, i = [1, \ldots, 4] \).

**Proof** see Appendix D.

These requirements constitute the traditional limits of using a polynomial utility function\(^{19}\) to represent individual preferences (Levy, 1969 and Tsiang, 1972). We can note, however, that, contrary to the quadratic one, the quartic specification is compatible – for some values of parameters – with the five desirable properties of utility functions: non-satiation, risk aversion, decreasing absolute risk aversion, decreasing absolute prudence and constant or increasing relative risk aversion (Pratt, 1964; Arrow, 1970 and Kimball, 1990 and 1993).

The theoretical conditions of an exact mean–variance–skewness–kurtosis decision criterion presented, in the next section we consider the conditions under which the mean–variance–skewness–kurtosis analysis can provide a satisfactory approximation rather than an exact solution of the expected utility optimisation problem.

1.4 EXPECTED UTILITY AS AN APPROXIMATING FUNCTION OF THE FIRST FOUR MOMENTS

While it is possible, by assuming specific parametric return distributions or utility functions to transform the expected utility principle into a mean–variance–skewness–kurtosis analysis, academics generally prefer to explore the conditions under which a fourth-order Taylor series expansion of the utility function can provide an accurate approximation of the investor’s objective function. However, since the conventional utility theory does not generally translate into a simple comparison of the first \( N \) moments, this alternative approach is in fact as restrictive as the previous one. Indeed, additional preference and distributional restrictions are needed to ensure that a quartic approximation of the expected utility provides an accurate and consistent local approximation of the individual objective function.

To address this issue, we consider the set of utility functions that display hyperbolic absolute risk aversion (HARA)\(^{20}\) and belong to the \( D_4 \) class. These utility functions are defined as:

\[
U(R) = \frac{\gamma}{(1 - \gamma)} \left( b + \frac{a}{\gamma} R \right)^{(1 - \gamma)}
\]

with:

\[
\begin{align*}
&b + \frac{a}{R} > 0 \\
&\frac{1}{\gamma} > -\frac{1}{2}
\end{align*}
\]

where, if \( \gamma \) is a negative integer then \( \gamma \leq -3 \) and \( \gamma \neq 0 \) otherwise, \( a > 0 \) and \( b \geq 0 \). It is easy to check that (1.26) satisfies the fourth-order stochastic dominance requirements, since:

\[
\begin{align*}
U^{(1)}(R) &= a \left( b + \frac{a}{\gamma} R \right)^{-\gamma} (> 0) \\
U^{(2)}(R) &= -a^2 \left( b + \frac{a}{\gamma} R \right)^{-1(\gamma+1)} (< 0) \\
U^{(3)}(R) &= a^3 \left( \frac{\gamma + 1}{\gamma} \right) \left( b + \frac{a}{\gamma} R \right)^{-1(\gamma+2)} (> 0) \\
U^{(4)}(R) &= -a^4 \frac{(\gamma + 1)(\gamma + 2)}{\gamma^2} \left( b + \frac{a}{\gamma} R \right)^{-1(\gamma+3)} (< 0)
\end{align*}
\]

\(^{20}\) For a study of the general HARA utility class, see Feigenbaum (2003) and Gollier (2004).
The HARA class relevant for the fourth-order stochastic dominance subsumes most of the popular functional forms of utility used in finance, including the constant absolute risk aversion (CARA) class, the constant relative risk aversion (CRRA) class and some functions belonging to the subclass of quartic utility and higher-degree polynomial utility functions.

The CRRA power and logarithmic utility functions can be obtained from (1.26) by selecting $\gamma > 0$ and $b := 0$, that is:

$$U(R) = \begin{cases} \frac{1}{(1-\gamma)} R^{(1-\gamma)} & \text{if } \gamma \neq 1 \\ \ln(R) & \text{if } \gamma = 1 \end{cases}$$  

(1.28)

where $\gamma$ is the constant investor’s relative risk-aversion coefficient. As for any HARA utility function, with $\gamma > 0$, the CRRA utility functions are also characterised by a decreasing absolute risk aversion with respect to the gross rate of return $R$.

If $\gamma := +\infty$ and $b := 1$, we obtain the (CARA) negative exponential utility function, that is:

$$U(R) = -\exp(-aR)$$  

(1.29)

where $a$ measures the constant investor’s absolute risk aversion.

For $\gamma := -(N-1)$ and $b > 0$, (1.26) reduces to a polynomial utility function of degree $N$, that is:

$$U(R) = \frac{(1-N)}{N} \left( \frac{a}{1-N} R + b \right)^N$$  

(1.30)

where $N \in \mathbb{IN}^* - \{1, 2, 3\}$ with $\gamma \leq -3$. This last specification covers only a subset of the quartic and higher-order polynomial utility functions (those with increasing absolute risk aversion) since HARA utility functions only have three free parameters, whereas a polynomial of degree $N$, with $N \geq 4$, has $(N + 1)$ parameters.

The quartic utility functions obtained from (1.26) by selecting $\gamma := -3$ and $b > 0$, have the following general form:

$$U(R) = R - \frac{3}{2} c R^2 + c^2 R^3 - \left( \frac{c}{2} \right)^3 R^4$$  

(1.31)

with:

$$c = 3 \left( \frac{b}{a} \right)$$

where $c > 0$.

If the gross rate of return belongs to the interval of absolute of convergence of the Taylor series expansion of the HARA utility function (1.26) around the expected gross rate of return, the investor’s utility can be expressed as:

$$U(R) = \sum_{n=0}^{N} \frac{1}{n!} (1-\gamma)^{-(n-1)} \left[ \prod_{i=0}^{n-1} \frac{(1-\gamma-i)}{(1-\gamma) \gamma^{(n-1)}} \right] (a)^n$$  

(1.32)

$$\times \left[ b + \frac{a}{\gamma} E(R) \right]^{-\gamma(n-1)} \left[ R - E(R) \right]^n$$

\(^{21\text{ HARA utility functions are analytic functions.}}\)
with:

\[ |R - E(R)| < \zeta \]

where \( \zeta \) is the positive radius of absolute convergence of the Taylor series expansion of \( U(.) \) around \( E(R) \) defined in (1.5), which is equal here to:

\[ \zeta = |-\gamma| \left( \frac{b}{a} + \frac{E(R)}{\gamma} \right) \]

and \( N \in IN \).

Provided the existence of the kurtosis and supposing that the investment probability distribution is uniquely determined by its moments, taking the limit of \( N \) towards infinity and the expected value on both sides in (1.32) leads to:

\[
E[U(R)] = \sum_{n=0}^{\infty} \frac{1}{n} (1 - \gamma)^{-1} \gamma^{-(n-1)} \left[ \prod_{i=0}^{n-1} \frac{(1 - \gamma - i)}{(1 - \gamma) \gamma^{(n-1)}} \right] (a)^n \\
\times \left[ b + \frac{a}{\gamma} E(R) \right]^{-(\gamma+n-1)} \cdot E\{[R - E(R)]^n\}
\]

with:

\[ |R - E(R)| \leq \zeta^* \]

where \( \zeta^* \in ]0, \zeta[ \) and \( \zeta \) is defined as in (1.5).

Focusing on terms up to the fourth, the expected utility can then be approximated by the following four-moment function\textsuperscript{22}:

\[
E[U(R)] \approx \frac{\gamma}{1 - \gamma} \left[ b + \frac{a}{\gamma} E(R) \right]^{(1-\gamma)} - \frac{a^2}{2} \left[ b + \frac{a}{\gamma} E(R) \right]^{-(\gamma+1)} \sigma^2(R) \\
+ \frac{a^3}{3!} \frac{(\gamma+1)}{\gamma} \left[ b + \frac{a}{\gamma} E(R) \right]^{-(\gamma+2)} s^3(R) \\
- \frac{a^4}{4!} \frac{(\gamma+1)(\gamma+2)}{\gamma^2} \left[ b + \frac{a}{\gamma} E(R) \right]^{-(\gamma+3)} \kappa^4(R)
\]

with (using previous notation):

\[ |R - E(R)| \leq \zeta^* \]

where \( \gamma \in IR^+ \), \( a > 0 \) and \( b \geq 0 \); \( \sigma^2(R) \), \( m^3(R) \), \( \kappa^4(R) \) are respectively the second, the third and the fourth centred moment of the returns \( R \), \( \zeta^* \in ]0, \zeta[ \) with \( \zeta \) defined as in (1.5).

This expression is consistent with our earlier comments regarding investors’ preference direction for higher moments, since the expected utility (1.34) depends positively on the expected return and skewness, and negatively on the variance and kurtosis, so that a positive skew in the returns distribution and less kurtosis lead to higher expected utility.

\textsuperscript{22} For quartic HARA utility functions, i.e. \( \gamma := -3 \) and \( b > 0 \), the Taylor series expansion of (1.26) leads to an equality in (1.34).
For the CARA and CRRA\textsuperscript{23} class preference specifications, the fourth-order Taylor approximation of the expected utility leads to the following analytical expressions (see Tsian, 1972; Hwang and Satchell, 1999; Guidolin and Timmermann, 2005a and 2005b and Jondeau and Rockinger, 2003b, 2005 and 2006):

\[
\begin{align*}
E \left[ (1 - \gamma)^{-1} R^{(1-\gamma)} \right] &\approx (1 - \gamma)^{-1} E (R)^{(1-\gamma)} - \frac{\gamma}{2} E (R)^{-(\gamma+1)} \sigma^2 (R) \\
&+ \frac{\gamma (\gamma + 1)}{3!} E (R)^{-(\gamma+2)} s^3 (R) - \frac{\gamma (\gamma + 1) (\gamma + 2)}{4!} \times E (R)^{-(\gamma+3)} \kappa^4 (R) \\
E [\ln (R)] &\approx \ln [E (R)] - \frac{1}{2} E (R)^{-2} \sigma^2 (R) + \frac{2}{3!} E (R)^{-3} s^3 (R) \\
&- \frac{6}{4!} E (R)^{-4} \kappa^4 (R) \\
E [- \exp (-aR)] &\approx - \exp [-aE (R)] \times \left[ 1 + \frac{a^2}{2} \sigma^2 (R) - \frac{a^3}{3!} s^3 (R) + \frac{a^4}{4!} \kappa^4 (R) \right]
\end{align*}
\]

(1.35)

with (using previous notation):

\[|R - E (R)| \leq \xi^* < \zeta\]

where \(\xi^* \in [0, \zeta]\) and \(\zeta\) is defined as in (1.5).

Through a fourth-order Taylor approximation, the investor’s decision problem under uncertainty can be simplified, but additional restrictions on the individual preferences and asset return distributions are required for guaranteeing the theoretical validity and practical interest of such an approach. Indeed, besides the necessary restrictions for the uniform convergence of the Taylor series expansion of \(U(.)\) around \(E (R)\), extra distributional conditions are required to guarantee the smoothness of the convergence of the Taylor polynomial towards the investor’s utility so that the quartic objective function (1.34) will perform uniformly better than the quadratic one. From the risk aversion property of the HARA utility functions, we get the following theorem.

**Theorem 6** A necessary condition\textsuperscript{24} for a fourth-order Taylor expansion of a HARA utility function of the class \(D_\alpha\), around \(E (R)\), to lead to a better expected utility approximation than a second-order one is that, \(\forall n \in \mathbb{N}^*:\)

\[
[(2n + 1)]^{-1} \times U^{(2n+1)} [E (R)] \times E \left\{ [R - E (R)]^{2n+1} \right\} \\
< - [(2n + 2)]^{-1} \times U^{(2n+2)} [E (R)] \times E \left\{ [R - E (R)]^{2n+2} \right\}
\]

(1.36)

\textsuperscript{23} Despite its prominence in the finance field, the usefulness of the CRRA class of utility functions for asset allocation with higher moments is limited due to the small responsiveness of CRRA investors to skewness and tail events (see Chen, 2003; Jondeau and Rockinger, 2003b and Cremers \textit{et al.}, 2005).

\textsuperscript{24} This condition is more general than the one proposed by Berényi (2001) since it is valid for any analytic utility function relevant for the fourth-order stochastic dominance and any probability distribution with finite variance and uniquely determined by the sequence of its moments.
with (using previous notation):

\[
U(n)[E(R)] = (1 - \gamma)^{-1} \gamma^{-(n-1)} \left[ \prod_{i=0}^{n-1} \frac{(1 - \gamma - i)}{(1 - \gamma) \gamma^{(n-1)}} \right] \times (a)^n \left[ \frac{a}{\gamma} E(R) + b \right]^{-(\gamma+n-1)}
\]

and:

\[
|R - E(R)| \leq \xi^*\]

where \(\xi^* \in ]0, \xi[\) and \(\xi, a, b\) and \(\gamma\) are defined respectively in (1.5) and (1.26).

**Proof** see Appendix E.

Another approach is to consider that the relative risk borne by investors, defined as the ratio of standard deviation by mean return, is so large that we cannot neglect it. In this case, Tsiang (1972) shows that, for most of the utility functions that belong to the HARA class, a suitable interval on which the relative risk is defined can be found, such that a quartic objective function provides a better approximation of the expected utility than a quadratic function. This approach is, however, less satisfactory than the last one, since it does not provide \emph{a priori} any clue regarding the limits of validity of the mean–variance–skewness–kurtosis decision criterion.

Following Samuelson (1970), it can also be shown that the speed of convergence of the Taylor series expansion towards the expected utility increases with the compactness of the portfolio return distribution and the length of the trading interval. That is, the smaller the trading interval and the absolute risk borne by individuals are, the less terms are needed in (1.34) to achieve an acceptable result.

Furthermore, since the conventional utility theory does not generally translate into a simple comparison of the first \(N\) moments, supplementary preference and distributional restrictions are needed to ensure that the quartic approximation (1.34) preserves individual preference ranking.

**Theorem 7** A necessary condition for the mean–variance–skewness–kurtosis function (1.34) to lead exactly to the same preference ordering as the expected utility criterion for an investor with a HARA utility of the \(D_4\) class is that the absolute risk aversion is decreasing \(\gamma > 0\), asset return distributions are negatively skewed and odd (even) higher-order moments are positive nonlinear functions of skewness (kurtosis), that is:\(^{25}\) \(\forall (i, n) \in (IN^+)^2:\)

\[
\begin{align*}
m^{2n+1}(R_i) &= (p_{2n+1})^{n+1} \left[ s^3(R_i) \right]^{\frac{2n+1}{3}} \\
m^{2n+2}(R_i) &= (q_{2n+2})^{n+\frac{3}{2}} \left[ \kappa^4(R_i) \right]^{\frac{n+\frac{3}{2}}{2}}
\end{align*}
\]

\(^{25}\) Similar statistical restrictions have been considered to simplify the mathematics of the mean–variance–skewness and mean–variance–kurtosis efficient frontiers (see Athayde and Flôres, 2002, 2003 and 2004).
Theoretical Foundations of Models with Higher-order Moments

with:

\[
\begin{align*}
p_3 &= q_4 = 1 \\
p_{2n+1} &> 0 \\
q_{2n+2} &> 0 \\
s^3(R_i) &< 0
\end{align*}
\]

and:

\[|R_i - E(R_i)| \leq \xi_i^*
\]

where \(\xi_i^* \in [0, \xi_i]\) with \(\xi_i\) the radius of absolute convergence of the Taylor series expansion of \(U(.)\) around \(E(R_i)\), \(i = [1, \ldots, N]\) and \((p_{2n+1} \times q_{2n+2}) = (IR_i^*)^2\).

**Proof** see Appendix F.

Nevertheless, whatever the truncation order chosen in (1.33), the accuracy of the approximation of the expected utility must be determined empirically (see Hlawitschka, 1994).

It has been shown on using different data, utility and parameter sets that a second-order Taylor expansion is an accurate approximation of the expected utility or the value function. For instance, using, respectively, annual returns on 149 US mutual funds and monthly returns on 97 randomly selected individual stocks from the CRSP database, Levy and Markowitz (1979) and Markowitz (1991) show that mean–variance portfolio rankings are highly correlated with those of the expected utility for most of the HARA utility functions and absolute risk-aversion levels. Moreover, using annual returns on 20 randomly selected stocks from the CRSP database, Kroll et al. (1984) find that mean–variance optimal portfolios are close to the ones obtained from the maximisation of the expected utility function when short-sales are forbidden, whatever leverage levels are considered. Similar results are obtained by Pulley (1981 and 1983), Reid and Tew (1987), Rafsnider et al. (1992), Ederington (1995) and Fung and Hsieh (1999) on different settings. Simaan (1997) also suggests that the opportunity cost of the mean–variance investment strategy is empirically irrelevant when the opportunity set includes a riskless asset, and very small in the absence of a riskless asset. Amilon (2001) and Cremers et al. (2004) extend the previous studies when short-sales are allowed. Working with monthly data on 120 Swedish stocks and five different families of utility functions, Amilon (2001) shows the opportunity cost of the mean–variance strategy remains relatively small for most of the investors in the presence of limited short-selling and option holding. Using monthly data on five different asset classes, Cremers et al. (2004) reach the same conclusions when the utility functions of the investors are logarithmic and the estimation risk is taken into account in the portfolio optimisation process.

Other recent empirical studies also show, however, that a fourth-order Taylor series expansion can improve significantly the quality of the investor’s expected utility approximation.

\[26\] Nonetheless, working with bootstrapped quarterly returns on 138 US mutual funds, Ederington (1995) finds that for strongly risk-averse investors, a Taylor series expansion based on the first four moments approximates the expected utility better than a Taylor series based on the first two moments.
For instance, working with monthly data on 54 hedge funds from the TASS database, Berényi (2002 and 2004) reports an increase of the correlation level between performance-based portfolio orderings and expected utility portfolio rankings when the considered performance measure takes into account the effect of the higher-order moments for various leverage levels and utility specifications. Jondeau and Rockinger (2003b and 2006) show that the mean–variance–skewness–kurtosis portfolio selection criterion leads to more realistic risky asset allocations than the mean–variance approach for some US stocks at a weekly frequency and emerging markets at a monthly frequency when investors have CARA or CRRA preferences and are short-sale constrained. Considering the tactical asset allocation of a US CRRA investor with holding periods ranging from one month to one year, Brandt et al. (2005) also find that approximated asset allocations obtained through the optimisation of a quartic objective function are closer to the true asset allocations than the ones obtained through the maximisation of a quadratic function and lead to lower certainty equivalent return losses than the second-order approximation, irrespective of the risk-aversion level and the investment horizon considered. The importance of higher moments for tactical asset allocation increases with the holding period and decreases with the level of the relative risk-aversion coefficient. Moreover, using ten years of monthly data on 62 hedge funds from the CIDSM database, Cremers et al. (2005) show that the mean–variance approach results in significant utility losses and unrealistic asset allocations for investors with bilinear or S-shaped value functions.

1.5 CONCLUSION

In this chapter we derive the theoretical foundations of multi-moment asset allocation and pricing models in an expected utility framework. We recall first the main hypotheses that are necessary to link the preference function with the centred moments of the unconditional portfolio return distribution. We then develop a quartic utility specification to obtain an exact mean–variance–skewness–kurtosis decision criterion. We also present the behavioural and distributional conditions under which the expected utility can be approximated by a fourth-order Taylor series expansion.

Our main conclusion is that, despite its widespread use in multi-moment asset allocation (see, for instance, Guidolin and Timmermann, 2005a and 2005b; Brandt et al., 2005 and Jondeau and Rockinger, 2003b, 2005 and 2006b) and capital and consumption asset pricing models (see, for instance, Kraus and Litzenberger, 1980; Fang and Lai, 1997; Hwang and Satchell, 1999; Dittmar, 2002 and Semenov, 2004), the Taylor series approach displays no general theoretical superiority over a polynomial utility specification to justify a moment-based decision criterion. Indeed, extra restrictions on the risky asset return distributions are required to ensure that a fourth-order Taylor series expansion preserves the preference ranking, while in the quartic case the opportunity set and utility parameters must be severely restricted to satisfy the five desirable properties of utility functions (see Pratt, 1964; Arrow, 1970 and Kimball, 1990). Moreover, additional requirements are necessary to guarantee the convergence of the (in)finite-order Taylor series expansions.

27 Patton (2001 and 2004) finds similar results in a conditional setting for small and large capitalisation US stock indices with monthly data and with or without short-sales.

28 Jondeau and Rockinger (2003b) investigate a similar risk-aversion effect when a risk-free asset exists.
Thus, the introduction of the third- and fourth-order centred moment in a portfolio selection criterion is theoretically justifiable when the utility function is quartic, and when the support of the portfolio distribution is well restricted or when individuals exhibit decreasing HARA utility functions, the time interval between actions and consequences is small but finite, and all the odd and even centred moments of investment returns can be expressed respectively as positive nonlinear functions of the (negative) skewness and the kurtosis. Under these conditions, it is then theoretically possible to derive a multi-moment asset pricing relation (see, for instance, Chapter 6).

**APPENDIX A**

**Theorem 1** A sufficient condition for a Taylor series expansion of an infinitely often differentiable utility function $U(.)$ around the expected gross rate of return $E (R)$ to converge absolutely is that the set of realisations of the random variable $R$ belongs to the open interval $J$ defined by:

$$|R - E (R)| < \zeta$$

with:

$$\zeta = \lim_{N\rightarrow\infty} \left| \frac{(N + 1)! U^{(N)} [E (R)]}{N! U^{(N+1)} [E (R)]} \right|$$

where $\zeta$ is a positive constant corresponding to the radius of convergence of the Taylor series expansion of $U(.)$ around $E (R)$ and $N \in \mathbb{N}$.

**Proof** Let $P_N [E (R)]$ be the Taylor approximation of order $N$ around $E (R)$ of an arbitrarily often differentiable utility function $U(.)$ on a subset $I$ of $\mathbb{R}$, that is:

$$P_N [E (R)] = \sum_{n=0}^{N} \frac{1}{n!} U^{(n)} [E (R)] [R - E (R)]^n \quad (1.38)$$

According to the quotient test, the Taylor polynomial (1.38) converges if:

$$q = \lim_{N\rightarrow\infty} \left| \frac{N! U^{(N+1)} [E (R)]}{(N + 1)! U^{(N)} [E (R)]} \times \frac{[R - E (R)]^{N+1}}{[R - E (R)]^N} \right| \quad (1.39)$$

$$= \lim_{N\rightarrow\infty} \left| \frac{N! U^{(N+1)} [E (R)]}{(N + 1)! U^{(N)} [E (R)]} \right| |R - E (R)| < 1$$

That is, the Taylor series expansion of $U(.)$ around the expected gross rate of return $E (R)$ is absolutely convergent when:

$$|R - E (R)| < \lim_{N\rightarrow\infty} \left| \frac{(N + 1)! U^{(N)} [E (R)]}{N! U^{(N+1)} [E (R)]} \right| \quad (1.40)$$
APPENDIX B

Theorem 2 A sufficient condition for a Taylor series expansion of an infinitely often differentiable utility function $U(.)$ around the expected gross rate of return $E(R)$ with a positive radius of convergence $\zeta$, to converge uniformly is that the set of realisations of the random variable $R$ remains in the closed interval $J^*$ defined as:

$$|R - E(R)| \leq \zeta^*$$

$\forall \zeta^* \in ]0, \zeta[$.

Proof Consider the Taylor approximation of order $N$ of an infinitely often differentiable utility function $U(.)$ defined on an open interval $J$ such that:

$$P_N[E(R)] = \sum_{n=0}^{N} \frac{1}{n!} U^{(n)} [E(R)] [R - E(R)]^n$$

where $N \in \mathbb{N}^*$ and $J = ]E(R) - \zeta, E(R) + \zeta]$ with $\zeta$ the radius of convergence of the Taylor series expansion of $U(.)$ around $E(R)$, e.g.:

$$\zeta = \lim_{N \to \infty} \left| \frac{(N + 1)! U^{(N)} [E(R)]}{N! U^{(N+1)} [E(R)]} \right|$$

For any $R \in [E(R) - \zeta^*, E(R) + \zeta^*]$, we must have:

$$|P[E(R)] - P_N[E(R)]| = \left| \sum_{n=N+1}^{\infty} \frac{1}{n!} U^{(n)} [E(R)] [R - E(R)]^n \right|
\leq \sum_{n=N+1}^{\infty} \left| \frac{1}{n!} U^{(n)} [E(R)] [R - E(R)]^n \right|
\leq \sum_{n=N+1}^{\infty} \left| \frac{1}{n!} U^{(n)} [E(R)] [\zeta^* - E(R)]^n \right|$$

with:

$$P[E(R)] = \sum_{n=0}^{\infty} \frac{1}{n!} U^{(n)} [E(R)] [R - E(R)]^n$$

where $\zeta^* \in ]0, \zeta[$.

That is:

$$\text{Sup}_{R \in [E(R) - \zeta^*, E(R) + \zeta^*]} \{|P[E(R)] - P_N[E(R)]|\} \leq \sum_{n=N+1}^{\infty} \left| \frac{1}{n!} U^{(n)} [E(R)] [\zeta^* - E(R)]^n \right|$$

Since the infinite-order Taylor series expansion of $U(.)$ around $E(R)$ is absolutely convergent on $J$, we get:

$$\lim_{N \to \infty} \sum_{n=N+1}^{\infty} \left| \frac{1}{n!} U^{(n)} [E(R)] [\zeta^* - E(R)]^n \right| = 0$$

(1.44)
We can thus make \(|P(\cdot) - P_N(\cdot)|\) as small as possible, \(\forall R \in [E(R) - \zeta^*, E(R) + \zeta^*]\), by choosing \(N\) sufficiently large and independent of \(R\). So that the Taylor series expansion of \(U(\cdot)\) around \(E(R)\) converges uniformly\(^{29}\) on the interval \([E(R) - \zeta^*, E(R) + \zeta^*]\)

\[\text{APPENDIX C}\]

\textbf{Theorem 3} \textit{The necessary and sufficient conditions for a quartic utility function \(U(\cdot)\) to belong to the class \(D_4\) of utility functions relevant for the fourth-order stochastic dominance are:}

\[U^{(1)}(\cdot) > 0 \iff \begin{cases} a_1 > 0 \\ \left( -\frac{a_3^2}{16a_4^2} + \frac{a_2}{6a_4} \right)^3 + \left( \frac{a_3^2}{16a_4^2} - \frac{a_2a_3}{16a_4^2} + \frac{a_1}{8a_4} \right)^2 > 0 \\ R < -\frac{a_3}{4a_4} + \frac{K + (9a_3^2 - 24a_4a_2)K^{-1}}{12a_4} \end{cases} \]

\[U^{(2)}(\cdot) < 0 \iff \begin{cases} 0 > a_2 \geq \left( \frac{3a_3^2}{8a_4} \right) \\ R < -\frac{a_3}{4a_4} + \frac{\sqrt{9a_3^2 - 24a_4a_2}}{12a_4} \text{ and } R > -\frac{a_3}{4a_4} - \frac{\sqrt{9a_3^2 - 24a_4a_2}}{12a_4} \text{ or } a_2 \leq \left( \frac{3a_3^2}{8a_4} \right) \end{cases} \]

\[U^{(3)}(\cdot) > 0 \iff \begin{cases} a_3 > 0 \\ R < -\left( \frac{a_3}{4a_4} \right) \end{cases} \]

\[U^{(4)}(\cdot) < 0 \iff a_4 < 0 \]

with:

\[K = \left( \frac{A + \sqrt{-108(3a_3^2 - 8a_4a_2)^3 + A^2}}{2} \right)^{1/3} \]

and:

\[A = (-54a_3^3 - 432a_2^2a_4 + 216a_4a_3a_2) \]

\(^{29}\) An infinite series \(\sum_{n=0}^{\infty} a_n x^n\) is said to converge uniformly on an interval \(J^*\) if, for each \(\varepsilon > 0\), we can find a positive integer \(N_0\) such that, \(\forall N \geq N_0\) and \(\forall x \in J^*\):

\[\left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n x^n \right| \leq \varepsilon\]

This is equivalent to saying that \(\{\sum_{n=0}^{\infty} a_n x^n\}\) converges uniformly on an interval \(J^*\) if, for each \(\varepsilon > 0\), we can find a positive integer \(N_0\) such that, \(\forall N \geq N_0\):

\[\text{Sup}_{x \in J^*} \left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{N} a_n x^n \right| \leq \varepsilon\]
where \( U^{(1)}(\cdot) \), \( U^{(2)}(\cdot) \), \( U^{(3)}(\cdot) \) and \( U^{(4)}(\cdot) \) are the first four partial derivatives of \( U(\cdot) \) and \( a_i \in IR^* \) with \( i = [1, \ldots, 4] \).

**Proof** Consider the following general quartic utility function:

\[
U(R) = a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4
\]  
(1.45)

with \( a_i \in IR^*, i = [1, \ldots, 4] \).

The first, second, third and fourth derivatives of (1.45) are given, respectively, by:

\[
\begin{align*}
U^{(1)}(R) &= a_1 + 2a_2 R + 3a_3 R^2 + 4a_4 R^3 \\
U^{(2)}(R) &= 2a_2 + 6a_3 R + 12a_4 R^2 \\
U^{(3)}(R) &= 6a_3 + 24a_4 R \\
U^{(4)}(R) &= 24a_4
\end{align*}
\]  
(1.46)

Imposing \( U^{(4)}(\cdot) < 0 \) and \( U^{(3)}(\cdot) > 0 \) yields the following set of requirements:

\[
\begin{align*}
a_3 &> 0 \\
R &< - \left( \frac{a_3}{4a_4} \right) \\
a_4 &< 0
\end{align*}
\]  
(1.47)

Depending on the sign of the discriminant of the quadratic expression of \( U^{(2)}(\cdot) \) in (1.46), the risk-aversion property \( U^{(2)}(\cdot) < 0 \) leads to:

\[
\begin{align*}
0 &> a_2 \geq \left( \frac{3a_3^2}{8a_4} \right) \\
R &< - \frac{a_3}{4a_4} + \frac{\sqrt{9a_3^2 - 24a_4a_2}}{12a_4} \quad \text{and} \quad R > - \frac{a_3}{4a_4} - \frac{\sqrt{9a_3^2 - 24a_4a_2}}{12a_4}
\end{align*}
\]  
(1.48)

From the non-satiation property, \( U^{(1)}(\cdot) < 0 \), we obtain the final system of parameter restrictions, that is:

\[
\begin{align*}
a_1 &> 0 \\
\left( a_1^2 + a_4 \right)^3 + \left( \frac{a_3^2}{16a_4} - a_2 a_3 + \frac{a_1}{8a_4} \right)^2 &> 0 \\
R &< - \frac{a_3}{4a_4} + \frac{K + (9a_3^2 - 24a_4a_2) K^{-1}}{12a_4}
\end{align*}
\]  
(1.49)

with:

\[K = \left( \frac{A + \sqrt{-108 (3a_3^2 - 8a_4a_2)^3 + A^2}}{2} \right)^{1/3}\]
Il: \[ A = \left( -54a_3^3 - 432a_3^2a_1 + 216a_3 a_3 a_2 \right). \]

Combining (1.47), (1.48) and (1.49) leads to the desired result. 

**APPENDIX D**

**Theorem 4** The necessary and sufficient conditions for a quartic utility function \( U(.) \) to exhibit a decreasing absolute risk aversion (DARA), a decreasing absolute prudence (DAP) and a constant or increasing relative risk aversion (CRRA or IRRA) with respect to the gross rate of return \( R \) are given by:

\[
\begin{align*}
8a_4^2R^4 + 8a_4a_3R^3 + 3a_3^2R^2 + 2(a_3a_2 - 2a_4a_1)R + \frac{2}{3}a_2^2 - a_3a_1 &< 0 \\
\left\{ \begin{array}{l}
\frac{a_3}{4a_4} + \frac{\sqrt{-9a_3^3 + 24a_4a_2}}{12a_4} - \frac{\sqrt{-9a_3^3 + 24a_4a_2}}{12a_4} < R < -\frac{a_3}{4a_4} \\
a_2 < \left( \frac{3a_3^2}{8a_4} \right)
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
&120(a_4)^2R^4 + 120a_3a_4R^3 + 3 \left[ 9(a_3)^2 + 16a_2a_4 \right]R^2 + 6(3a_2a_3 + 2a_4a_1)R + 3a_3a_1 + 2(a_2)^2 \geq \\
&(12a_4R^2 + 6a_3R + 2a_2)(4a_4R^4 + 3a_3R^3 + 2a_2R^2 + a_1R)^{-1}
\end{align*}
\]

where \( a_i \in IR^n, \) with \( i = [1, \ldots , 4] \).

**Proof** Imposing the decrease of absolute risk-aversion coefficient with respect to the gross rate of return, that is:

\[
\left[ -\frac{\partial U(R)}{\partial R} \frac{\partial^3 U(R)}{\partial R^3} + \left( \frac{\partial^2 U(R)}{\partial R^2} \right)^2 \right] < 0 \tag{1.50}
\]

we obtain, from the derivative expressions (1.46) of the quartic utility function, that the DARA property is verified when:

\[
8a_4^2R^4 + 8a_4a_3R^3 + 3a_3^2R^2 + 2(a_3a_2 - 2a_4a_1)R + \frac{2}{3}a_2^2 - a_3a_1 < 0 \tag{1.51}
\]

Imposing the decrease of absolute prudence coefficient with respect to the gross rate of return gives:

\[
\left[ -\frac{\partial^2 U(R)}{\partial R^2} \frac{\partial^4 U(R)}{\partial R^4} + \left( \frac{\partial^3 U(R)}{\partial R^3} \right)^2 \right] < 0 \tag{1.52}
\]
In the quartic case, this inequality translates into (see (1.46)):

\[ 24a_4^2R^2 + 12a_4a_3R + 3a_3^2 + 4a_4a_2 < 0 \] (1.53)

Depending on the sign of the discriminant in (1.53), we find that the DAP is achieved for quartic utility functions when:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{1}{4a_4} - \sqrt{\frac{-9a_2^2 + 24a_4a_2}{12a_4}} < R < -\frac{a_3}{4a_4} - \sqrt{\frac{-9a_2^2 + 24a_4a_2}{12a_4}} \\
a_2 < \left( \frac{3a_2^2}{8a_4} \right) \\
or a_2 > \left( \frac{3a_2^2}{8a_4} \right)
\end{array} \right.
\] (1.54)

Finally, the increase or constancy of relative risk-aversion coefficient with respect to the gross rate of return leads to the following restriction:

\[
R \times \left[ -\frac{\partial^2 U(R)}{\partial R^2} \frac{\partial^4 U(R)}{\partial R^4} + \left( \frac{\partial^3 U(R)}{\partial R^3} \right)^2 \right] - \frac{\partial^2 U(R)}{\partial R^2} \left( \frac{\partial U(R)}{\partial R} \right)^{-1} \geq 0
\] (1.55)

Using the derivative expressions (1.46), this property requires, for the fourth-order polynomial utility function, that:

\[
8a_4^2R^5 + 8a_4a_3R^4 + 3a_3^2R^3 - 2 \left[ a_4a_1 - 2a_3a_2 - 6(a_1 + 2a_2R + 3a_3R^2 + 4a_4R^3)^{-1}a_4 \right] R^2
\]

\[
- \left[ a_3a_1 - \frac{2}{3}a_2^2 + 6(a_1 + 2a_2R + 3a_3R^2 + 4a_4R^3)^{-1}a_3 \right] R
\]

\[
- 2a_2 \left( 4a_4R^3 + 3a_3R^2 + 2a_5R + a_4 \right)^{-1} \leq 0
\] (1.56)

\[\blacksquare\]

**APPENDIX E**

**Theorem 5** A necessary condition for a fourth-order Taylor expansion of a HARA utility function of Dₜ utility class around E(R) to lead to a better expected utility approximation than a second-order one is that, \( \forall n \in \mathbb{N}^+ \):

\[
[(2n + 1)!]^{-1} U^{(2n+1)}[E(R)] \times E \left\{ [R - E(R)]^{2n+1} \right\}
\]

\[
< - [(2n + 2)!]^{-1} U^{(2n+2)}[E(R)] \times E \left\{ [R - E(R)]^{2n+2} \right\}
\]

with (using previous notation):

\[
U^{(n)}[E(R)] = (1 - \gamma)^{-1} \gamma^{-(n-1)} \left[ \prod_{i=0}^{n-1} \frac{1 - \gamma - l}{(1 - \gamma)^{\gamma^{(n-1)}}} \right] \times (a)^n \left[ \frac{a}{\gamma} E(R) + b \right]^{-(\gamma^{n-1})}
\]
The risk aversion property entails, for any risky investment, that:

\[ E[U(R)] = U[E(R)] + \Phi[E(R)] \] 

(1.57)

with:

\[ \Phi[E(R)] = \sum_{n=2}^{\infty} \frac{1}{n!} U^{(n)}[E(R)] m^n(R) \]

and:

\[ |R - E(R)| \leq \zeta^* \]

where \( \zeta^* \in ]0,\zeta[\), is the positive radius of absolute convergence of the Taylor series expansion of \( U(.) \) around \( E(R) \).

If, moreover, the centred higher-order moments of the gross rate of return distribution, denoted \( m^n(R) \), verify \( \forall n \in IN^* \):

\[ [(2n + 1)!]^{-1} \times U^{2n+1}[E(R)] \times m^{2n+1}(R) < -[(2n + 2)!]^{-1} \times U^{2n+2}[E(R)] \times m^{2n+2}(R) \]

(1.58)

then, the \( n \)th order Taylor approximation of the expected utility of an agent with HARA-type preference must converge smoothly towards its objective function, since, \( \forall N \in IN^* \) and \( R \in [E(R) - \zeta^*, E(R) + \zeta^*] \):

\[ \frac{1}{2} U^{(2)}[E(R)] \sigma^2(R) + \sum_{n=3}^{2N+2} \frac{1}{n!} U^{(n)}[E(R)] m^n(R) < 0 \]

(1.59)

and:

\[ E[U(R)] < U[E(R)]. \]
It follows that the fourth-order Taylor series expansion of a HARA utility function which belongs to the $D_4$ class will lead to a better approximation of the expected utility criterion than the one obtained through a second-order Taylor series expansion, $\forall R \in [E(R) - \xi^*, E(R) + \xi^*$].

\section*{APPENDIX F}

\textbf{Theorem 6} A necessary condition for the mean–variance–skewness–kurtosis function (1.34) to lead exactly to the same preference ordering as the expected utility criterion for an investor with a HARA utility $U(.)$ of the $D_4$ utility class is that the absolute risk aversion is decreasing ($\gamma > 0$), asset return distributions are negatively skewed and odd (even) higher-order moments are positive nonlinear functions of skewness (kurtosis), that is, $\forall (i, n) \in (IN^*)^2$:

\begin{align*}
&\begin{cases}
m^{2n+1} (R_i) = (p_{2n+1})^{\frac{2n+1}{n+1}} [s^3 (R_i)]^{\frac{2n+1}{n+1}} \\
m^{2n+2} (R_i) = (q_{2n+2})^{\frac{n+1}{n+1}} [\kappa^4 (R_i)]^{\frac{n+1}{n+1}}
\end{cases}
\end{align*}

with:

\begin{align*}
&\begin{cases}
p_3 = q_4 = 1 \\
p_{2n+1} > 0 \\
q_{2n+2} > 0 \\
s^3 (R_i) < 0
\end{cases}
\end{align*}

and:

\[ |R_i - E(R_i)| \leq \xi^*_i \]

where $\xi^*_i \in ]0, \xi_i[ \text{ with } \xi_i \text{ the positive radius of absolute convergence of the Taylor series expansion of } U(.) \text{ around } E(R_i), i = [1, \ldots, N], \text{ and } (p_{2n+1} \times q_{2n+2}) = (IR^*_+)^2$.

\textbf{Proof} Consider a HARA utility function $U(.)$ of the $D_4$ class and two negatively skewed distributed random returns $R_1 \in [\xi^*_1 - E(R_1), \xi^*_1 + E(R_1)]$ and $R_2 \in [\xi^*_2 - E(R_2), \xi^*_2 + E(R_2)]$, with finite moments such that:

\begin{align*}
E(R_1) \geq E(R_2) \\
\sigma^2(R_1) \leq \sigma^2(R_2) \\
s^3(R_1) \geq s^3(R_2) \\
\kappa^4(R_1) \leq \kappa^4(R_2)
\end{align*}
with at least one strict inequality$^{32}$, and $\forall i \in [1, 2]$ and $\forall n \in \mathbb{N}^*:
\begin{align}
\begin{cases}
m_{2n+1}^2 (R_i) = (p_{2n+1})^{\frac{2n+1}{2}} \left[ s^3 (R_i) \right]^{\frac{2n+1}{6}} \\
m_{2n+2}^2 (R_i) = (q_{2n+2})^{\frac{2n+2}{2}} \left[ \kappa^4 (R_i) \right]^{\frac{n+1}{2}}
\end{cases}
\end{align}
(1.61)

with:
\[
\begin{cases}
p_3 = q_4 = 1 \\
p_{2n+1} > 0 \\
q_{2n+2} > 0 \\
s^3 (R_i) < 0
\end{cases}
\]

where $\zeta^*_i \in ]0, \zeta_i[ , \zeta_i$ is the positive radius of absolute convergence of the Taylor series expansion of $U(\cdot)$ around $E(R_i)$, $m^n (R_i) = E \left[ (R_i - E(R_i))^n \right]$, $i = [1, 2]$, and $(p_{2n+1} \times q_{2n+2}) = (IR^n)^2$.

For any decreasing HARA utility function $U(\cdot)$, there is a strict equivalence between the mean–variance–skewness–kurtosis preference ranking and the expected utility preference ordering, since:

\[
E \left[ U(R_1) \right] - E \left[ U(R_2) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ U^{(n)} \left[ E(R_1) \right] \times m^n (R_1) - U^{(n)} \left[ E(R_2) \right] \times m^n (R_2) \right\} > 0 \tag{1.62}
\]

with:
\[
\begin{cases}
(-1)^n U^{(n)} (R_i) < 0 \\
(-1)^n m^n (R_1) \leq (-1)^n m^n (R_2)
\end{cases}
\]

and:

\[|R_i - E(R_i)| \leq \zeta^*_i\]

where $i = [1, 2]$ and $n \in \mathbb{N}^* - \{1\}.$

\[\mathbf{\blacksquare}\]

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$^{32}$ So that, asset 1 with return $R_1$ is preferred to asset 2 with return $R_2$ for any investor with mean–variance–skewness–kurtosis preferences.
REFERENCES


