CHAPTER ONE

Localized Waves: A Historical and Scientific Introduction

ERASMO RECAMI
Università degli Studi di Bergamo, Bergamo, Italy, and INFN–Sezione di Milano, Milan, Italy

MICHEL ZAMBOÑI-RACHED
Centro de Ciências Naturais e Humanas, Universidade Federal do ABC, Santo André, SP, Brazil

HUGO E. HERNÁNDEZ-FIGUEROA
Universidade Estadual de Campinas, Campinas, SP, Brazil

In the first part of this introductory chapter, we present general and formal (simple) introductions to the ordinary Gaussian waves and to the Bessel waves, by explicitly separating the cases of the beams from the cases of the pulses; and, finally, an analogous introduction is presented for the localized waves (LW), pulses or beams. Always we stress the very different characteristics of the Gaussian with respect to the Bessel waves and to the LWs, showing the numerous and important properties of the latter w.r.t. the former ones: Properties that may find application in all fields in which an essential role is played by a wave-equation (like electromagnetism, optics, acoustics, seismology, geophysics, gravitation, elementary particle physics, etc.). In the second part of this chapter (namely, in its Appendix), we recall at first how, in the seventies and eighties, the geometrical methods of special relativity (SR) predicted—in the sense below specified—the existence of the most interesting LWs, i.e., of the X-shaped pulses. At last, in connection with the circumstance that the X-shaped waves

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Localized waves are endowed with superluminal group-velocities (as carefully discussed in the first part of this chapter), we mention briefly the various experimental sectors of physics in which superluminal motions seem to appear: In particular, a bird’s-eye view is presented of the experiments till now performed with evanescent waves (and/or tunneling photons), and with the “localized superluminal solutions” to the wave equations.

1.1 GENERAL INTRODUCTION

Diffraction and dispersion have long been known as phenomena that limit the application of (e.g., optical) beams or pulses. Diffraction is always present, affecting any waves that propagate in two- or three-dimensional media, even when homogeneous. Pulses and beams comprise waves traveling along different directions which produce gradual spatial broadening [6]. This effect is a limiting factor whenever a pulse is needed that maintains its transverse localization (e.g., in free-space communications [7], image forming [8], optical lithography [9,10], electromagnetic tweezers [11,12]).

Dispersion acts on pulses propagating in material media, causing mainly temporal broadening: an effect known to be due to the variation in refraction index with frequency, so that each spectral component of the pulse possesses a different phase velocity. This entails gradual temporal widening, which constitutes a limiting factor when a pulse is needed that maintains its time width (e.g., in communication systems [13]).

It is important, therefore, to develop any techniques able to reduce those phenomena. Localized waves, also known as nondiffracting waves, are indeed able to resist diffraction for a long distance in free space. Such solutions to the wave equations (and, in particular, to Maxwell’s equations, under weak hypotheses) were predicted theoretically long ago [14–17] (cf. also [18] and the Appendix to this chapter), constructed mathematically in more recent times [19,20] and soon after produced experimentally [21–23]. Today, localized waves are well established both theoretically and experimentally and are being used in innovative applications not only in vacuum but also in material (linear or nonlinear) media, showing to be able to resist also dispersion. As we mentioned, their potential applications are being explored intensively, always with surprising results, in such fields as acoustics, microwaves, and optics, and are also promising in mechanics, geophysics, and even in gravitational waves and elementary particle physics. Also worth noting are the possible applications of the “frozen waves,” discussed in Chapter 2, and the ones already obtained, for instance, in high-resolution ultrasound scanning of moving organs in the human body [24,25].

Restricting ourselves to electromagnetism, we cite present-day studies on electromagnetic tweezers [26–29], optical (or acoustic) scalpsels, optical guiding of atoms or (charged or neutral) corpuscles [30–32], optical litography [26,33], optical (or acoustic) images [34], communications in free space [19,35–37], remote optical alignment [38], and optical acceleration of charged corpuscles, among others.

Next we describe briefly the theory and applications of localized beams and pulses.

Localized (Nondiffracting) Beams The word beam refers to a monochromatic solution to a wave equation, with transverse localization of its field. To fix our ideas,
we refer explicitly to the optical case, but our considerations hold for any wave equation (vectorial, spinorial, scalar—in particular, for the acoustic case).

The most common type of optical beam is the Gaussian beam, whose transverse behavior is described by a Gaussian function. But all the common beams suffer a diffraction, which spoils the transverse shape of their field, widening it gradually during propagation. As an example, the transverse width of a Gaussian beam doubles when it travels a distance \( z_{\text{dif}} = \sqrt{3 \pi \Delta \rho_0^2 / \lambda_0} \), where \( \Delta \rho_0 \) is the beam initial width and \( \lambda_0 \) is its wavelength. One can verify that a Gaussian beam with an initial transverse aperture of the order of its wavelength will double its width after having traveled only a few wavelengths.

It was generally believed that the only wave devoid of diffraction was the plane wave, which does not undergo any transverse change, but some authors had shown that it is not the only one. For instance, in 1941, Stratton [15] obtained a monochromatic solution to the wave equation whose transverse shape was concentrated in the vicinity of its propagation axis and represented by a Bessel function. Such a solution, now called a Bessel beam, was not subject to diffraction, since no change in its transverse shape took place with time. Later, Courant and Hilbert [16] demonstrated how a large class of equations (including the wave equations) admit “nondistorted progressing waves” as solutions; and as early as 1915, Bateman [17] and others [39] showed the existence of soliton-like, wavelet-type solutions to Maxwell’s equations. But all such literature did not attract the attention it deserved. In Stratton’s work [15] this can be partially justified since the Bessel beam was endowed with infinite energy (as much as the plane waves). An interesting problem, therefore, was that of investigating what would happen to the ideal Bessel beam solution when truncated by a finite transverse aperture.

Not until 1987 did a heuristical answer came from an actual experiment, when Durnin et al. [40] showed that a realistic Bessel beam endowed with wavelength \( \lambda_0 = 0.6328 \, \mu \text{m} \) and central spot† \( \Delta \rho_0 = 59 \, \mu \text{m} \), passing through an aperture with radius \( R = 3.5 \, \text{mm} \), is able to travel about 85 cm keeping its transverse intensity shape approximately unchanged (in the region \( \rho < R \) surrounding its central peak). In other words, it was shown experimentally that the transverse intensity peak as well as the field surrounding it do not undergo any appreciable change in shape all along a large depth of field. By comparison, let us recall once more that a Gaussian beam with the same wavelength and with the central “spot”‡ \( \Delta \rho_0 = 59 \, \mu \text{m} \), when passing through an aperture with the radius \( R = 3.5 \, \text{mm} \), doubles its transverse width after 3 cm, and after 6 cm its intensity has already diminished by a factor of 10. Therefore, in the case considered, a Bessel beam can travel, approximately without deformation, a distance 28 times longer than that traveled by a Gaussian beam.

Such a remarkable property is due to the fact that, when diffracting, the transverse intensity fields (whose value decreases with increasing \( \rho \)), associated with the rings that constitute the (transverse) structure of the Bessel beam, end up reconstructing the beam itself all along a large depth of field. All this depends on the Bessel beam

† Let us define the central spot of a Bessel beam as the distance, along the propagation axis \( \rho = 0 \), at which occurs the first zero of the Bessel function, characterizing its transverse shape.

‡ In the case of a Gaussian beam, let us define its central spot as the distance along \( \rho = 0 \) at which its transverse intensity has decayed by the factor \( 1/e \).
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Let us stress that given a Bessel and a Gaussian beam, both with the same spot \( \Delta \rho_0 \) and passing through apertures with the same radius \( R \) in the plane \( z = 0 \) and with the same energy \( E \), the percentage of the total energy \( E \) contained inside the central peak region \( 0 \leq \rho \leq \Delta \rho_0 \) is smaller for a Bessel than for a Gaussian beam. This different energy distribution on the transverse plane is responsible for reconstruction of the Bessel beam central peak even at large distances from the source (and even for an obstacle smaller than the aperture [71,78,117]: a nice property also possessed by the localized pulses [117] we examine below).

It may be worth mentioning that most experiments carried on in this area have been performed rapidly and, often, using rather simple apparatus. For example, Durnin et al.’s setup, used to generate a Bessel beam, had a laser source, an annular slit, and a lens, as depicted in Fig. 1.1. In a sense, such apparatus produces what can be regarded as the cylindrically symmetric generalization of a couple of plane waves emitted at angles \( \theta \) and \( -\theta \) with respect to the \( z \)-direction (in which case the plane-wave intersection moves along \( z \) with the speed \( c/\sin \theta \)). Of course, these nondiffracting beams can also be generated by a conic lens (an axicon) (see, e.g., [34]), or by other means, such as holographic elements (cf., e.g., [38,44]).

As mentioned earlier, a lot of interesting applications of nondiffracting beams are being investigated in addition to Lu et al.’s work in acoustics. In the optical sector, one example is the use of Bessel beams as optical tweezers able to confine small particles or move them around. In such theoretical and application areas, a noticeable contribution is the one presented in [45,46,77], wherein, by suitable superposition of Bessel beams endowed with the same frequency but different longitudinal wave numbers, stationary fields have been constructed mathematically in closed form, which possess a high transverse localization and, more important, a longitudinal intensity shape that can be chosen freely inside a predetermined space interval \( 0 \leq z \leq L \). For instance, a high-intensity field, with a static envelope, can be created within a tiny region, with negligible intensity elsewhere. Chapter 2 deals with such “frozen waves.”
Localized (Nondiffracting) Pulses  As we have seen, the existence of nondiffractive (or localized) pulses was long predicted; see [16,17], and also [14,18], as well as more recent articles (e.g., [47,48]). Modern studies on nondiffractive pulses (confining ourselves to those that attracted more attention) developed somewhat independently of those on nondiffracting beams, even if both phenomena are part of the same sector of physics: localized waves.

In 1983, Brittingham [49] set forth a luminal ($V = c$) solution to the wave equation (more particularly, to Maxwell’s equations), which travels rigidly (i.e., without diffraction). The solution proposed [49] possessed infinite energy, however, and once more the problem arose of overcoming such a problem. A way out was first obtained by Sezginer [50], who showed how to construct finite-energy luminal pulses, which, however, do not propagate without distortion for an infinite distance, but, as expected, travel with constant speed and approximately without deformation for a certain (long) depth of field: much longer, in this case, too, than that of ordinary pulses such as the Gaussian pulses. In a series of subsequent papers [35,36,51–54], a simple theoretical method was developed, called bidirectional decomposition by those authors, for constructing a new series of nondiffracting luminal pulses.

Finally, at the beginning of the 1990s, Lu et al. [19,21] constructed, both mathematically and experimentally, new solutions to the wave equation in free space: namely, an X-shaped localized pulse, with the form predicted by the so-called extended special relativity [1,14]; for the connection between what Lu et al. called X-waves and extended relativity, see, for example, [18]; brief excerpts of that theory can also be found, for example, in [20,55–58]. Lu et al.’s solutions were continuous superpositions of Bessel beams with the same phase velocity (i.e., with the same axicon angle [1,19,20,59]; cf. Fig. 1.2) so they could keep their shape for long distances. Such X-shaped waves resulted in interesting and flexible localized solutions, even if their velocity $V$ is supersonic or superluminal ($V > c$), and have been studied in a number of papers. Actually, when the phase velocity does not depend on the
frequency, it is known that such a phase velocity becomes the group velocity! Remembering how a superposition of Bessel beams is generated (e.g., by a discrete or continuous set of annular slits or transducers), it is clear that the energy forming the X-waves coming from those rings travels at the ordinary speed $c$ of plane waves in the medium considered [20,60–62]. (Here, $c$, representing the velocity of the plane waves in the medium, is the speed of sound in the acoustic case, the speed of light in the electromagnetic case, and so on.) Nevertheless, the peak of the X-shaped waves is faster than $c$.

It is possible to generate (in addition to the “classic” X-wave produced by Lu et al. in 1992) infinite sets of new X-shaped waves, with their energy concentrated more and more in a spot corresponding to the vertex region [42]. It may therefore appear rather intriguing that such a spot (even if no violation of special relativity is obviously implied: all the results come from Maxwell’s equations or from the wave equations [73,74]) travels superluminally when the waves are electromagnetic. We shall call all the X-shaped waves superluminal even when, for example, the waves are acoustic. In Fig. 1.3, we illustrate the fact that if its vertex or central spot is located at $P_1$ at time $t_1$, it will reach position $P_2$ at time $t + \tau$, where $\tau = |P_2 - P_1|/V < |P_2 - P_1|/c$. We discuss all these points below.

Soon after having constructed their “classic” acoustic X-wave mathematically and experimentally, Lu et al. started applying them to ultrasonic scanning, obtaining, as we already said, very high quality images. Subsequently, in a 1996 e-print and report, Recami et al. (see, e.g., [20] and references therein) published the analogous X-shaped solutions to Maxwell’s equations: by constructing scalar superluminal localized solutions for each component of the Hertz potential. That showed, by the way, that localized solutions to scalar equation can also be used, under very weak conditions, for obtaining localized solutions to Maxwell’s equations (actually,
Ziolkowski et al. [43] had found something similar, which they called slingshot pulses, for the simple scalar case, but their solution had gone practically unnoticed. In 1997, Saari and Reivelt [22] announced, in an important paper, the production in the lab of an X-shaped wave in the optical realm, thus proving experimentally the existence of superluminal electromagnetic pulses. Three years later, in 2000, Mugnai et al. [23] produced, experimentally, superluminal X-shaped waves in the microwave region (their paper aroused various criticisms, to which the authors responded).

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Let us consider [5] the differential equation known as homogeneous wave equation: simple, but so important in acoustics, electromagnetism (microwaves, optics, etc.), geophysics, and even, as we said, gravitational waves and elementary particle physics:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(x, y, z; t) = 0. \tag{1.1}
\]

Let us write it in the cylindrical coordinates \((\rho, \phi, z)\) and, for simplicity’s sake, confine ourselves to axially symmetric solutions \(\psi(\rho, z; t)\). Then, Eq. (1.1) becomes

\[
\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(\rho, z; t) = 0. \tag{1.2}
\]

In free space, the solution \(\psi(\rho, z; t)\) can be written in terms of a Bessel–Fourier transform with respect to the variable \(\rho\), and two Fourier transforms with respect to variables \(z\) and \(t\):

\[
\psi(\rho, z, t) = \int_0^\infty \int_{-\infty}^{\infty} k_\rho J_0(k_\rho \rho) e^{i k_\rho z} e^{-i \omega t} \tilde{\psi}(k_\rho, k_z, \omega) dk_\rho dk_z d\omega, \tag{1.3}
\]

where \(J_0(\cdot)\) is an ordinary zero-order Bessel function and \(\tilde{\psi}(k_\rho, k_z, \omega)\) is the transform of \(\psi(\rho, z, t)\).

Substituting Eq. (1.3) into Eq. (1.2), one finds that the relation

\[
\frac{\omega^2}{c^2} = k_\rho^2 + k_z^2 \tag{1.4}
\]

among \(\omega, k_\rho, \) and \(k_z\) has to be satisfied. In this way, by using condition (1.4) in Eq. (1.3), any solutions to the wave equation (1.2) can be written

\[
\psi(\rho, z, t) = \int_0^{\omega/c} \int_{-\infty}^{\infty} k_\rho J_0(k_\rho \rho) e^{i z \sqrt{\omega^2/c^2 - k_\rho^2}} e^{-i \omega t} S(k_\rho, \omega) dk_\rho d\omega, \tag{1.5}
\]

where \(S(k_\rho, \omega)\) is the spectral function chosen.
The general integral solution (1.5) yields, for instance, (nonlocalized) Gaussian beams and pulses, to which we shall refer to illustrate the differences between localized waves and them.

**Gaussian Beam** A very common (nonlocalized) beam is the Gaussian beam [76], corresponding to the spectrum

\[ S(k_\rho, \omega) = 2a^2 e^{-a^2 k_z^2} \delta(\omega - \omega_0). \] (1.6)

In Eq. (1.6), \( a \) is a positive constant that will be shown to depend on the transverse aperture of the initial pulse.

Figure 1.4 illustrates the interpretation of integral solution (1.5) with spectral function (1.6) as a superposition of plane waves. From Fig. 1.4 one can easily realize that this case corresponds to plane waves propagating in all directions (always with \( k_z \geq 0 \)), the most intense being those directed along (positive) \( z \).

Notice that in the plane-wave case, \( \vec{k}_z \) is the longitudinal component of the wave vector, \( \vec{k} = \vec{k}_\rho + \vec{k}_z \), where \( \vec{k}_\rho = \vec{k}_x + \vec{k}_y \).

On substituting Eq. (1.6) into Eq. (1.5) and adopting the paraxial approximation, one obtains the Gaussian beam:

\[ \psi_{\text{Gauss}}(\rho, z, t) = \frac{2a^2 \exp[-\rho^2/4(a^2 + iz/2k_0)]}{2(a^2 + iz/2k_0)} e^{ik_0(z-ct)}, \] (1.7)

where \( k_0 = \omega_0/c \). We can verify that such a beam, which suffers transverse diffraction, doubles the initial width \( \Delta \rho_0 = 2a \) after having traveled the distance \( z_{\text{dif}} = \sqrt{3} k_0 \Delta \rho_0^2/2 \), called the *diffraction length*. The more concentrated a Gaussian beam happens to be, the more rapidly it gets spoiled.
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Gaussian Pulse The most common (nonlocalized) pulse is the Gaussian pulse, which is obtained from Eq. (1.5) using the spectrum [75]

\[
S(k_{\rho}, \omega) = \frac{2ba^2}{\sqrt{\pi}} e^{-a^2k_{\rho}^2} e^{-b^2(\omega - \omega_0)^2},
\]

(1.8)

where \(a\) and \(b\) are positive constants. Indeed, such a pulse is a superposition of Gaussian beams of different frequencies.

Now, on substituting Eq. (1.8) into Eq. (1.5), and again adopting the paraxial approximation, one gets the Gaussian pulse

\[
\psi(\rho, z, t) = \frac{a^2}{a^2 + iz/2k_0} \exp\left[\frac{-\rho^2}{4(a^2 + iz/2k_0)}\right] \exp\left[-(z - ct)^2/4c^2b^2\right],
\]

(1.9)

endowed with speed \(c\) and temporal width \(\Delta t = 2b\), and suffering a progressive enlargement of its transverse width so that its initial value is already doubled at position \(z_{\text{dif}} = \sqrt{3}k_0 \Delta \rho_0^2/2\), with \(\Delta \rho_0 = 2a\).

1.2.1 Localized Solutions

Finally, let’s look at the construction of the two most renowned localized waves: the Bessel beam and the ordinary X-shaped pulse [5].

It is interesting, first, to observe that, when superposing (axially symmetrical) solutions of the wave equation in vacuum, three spectral parameters \((\omega, k_{\rho}, k_z)\) come into play which have however to satisfy constraint (1.4), deriving from the wave equation itself. Consequently, only two of them are independent, and here we choose \(\omega\) and \(k_{\rho}\). Such freedom in choosing \(\omega\) and \(k_{\rho}\) was apparent in the spectral functions generating Gaussian beams and pulses, which are the product of two functions, one depending only on \(\omega\) and the other on \(k_{\rho}\).

We are going to see, moreover, that particular relations can be imposed between \(\omega\) and \(k_{\rho}\) (or analogously, between \(\omega\) and \(k_z\)) in order to get interesting and unexpected results, such as localized waves.

Bessel Beam Let us start by imposing a linear coupling between \(\omega\) and \(k_{\rho}\) (it could actually be shown [41] that it is the unique coupling leading to localized solutions). Let us consider the spectral function

\[
S(k_{\rho}, \omega) = \frac{\delta(k_{\rho} - (\omega/c) \sin \theta)}{k_{\rho}} \delta(\omega - \omega_0),
\]

(1.10)

which implies that \(k_{\rho} = (\omega \sin \theta)/c\), with \(0 \leq \theta \leq \pi/2\), a relation that can be regarded as a space–time coupling. Let us add that this linear constraint between \(\omega\) and \(k_{\rho}\),

\(^{\dagger}\)Elsewhere we chose \(\omega\) and \(k_z\).
FIGURE 1.5 An axially symmetric Bessel beam is created by the superposition of plane waves whose wave vectors lay on the surface of a cone having the propagation axis as its symmetry axis and angle equal to \( \theta \) (axicon angle).

Together with relation (1.4), yields \( k_z = (\omega \cos \theta)/c \). This is an important fact, since it has been shown elsewhere [42] that an ideal localized wave must contain a coupling of the type \( \omega = V k_z + b \), where \( V \) and \( b \) are arbitrary constants.

The interpretation of the integral function (1.5), this time with the spectrum (1.10), as a superposition of plane waves is visualized in Fig. 1.5, which shows that an axially symmetric Bessel beam is nothing but the result of the superposition of plane waves whose wave vectors lay on the surface of a cone having the propagation axis as its symmetry axis and an angle equal to \( \theta \), the axicon angle.

By inserting Eq. (1.10) into Eq. (1.5), one gets the mathematical expression of a Bessel beam:

\[
\psi(\rho,z,t) = J_0 \left( \frac{\omega_0}{c} \sin \theta \rho \right) \exp \left[ i \frac{\omega_0}{c} \cos \theta \left( z - \frac{c}{\cos \theta} t \right) \right]. \tag{1.11}
\]

This beam possesses phase velocity \( v_{ph} = c/\cos \theta \) and field transverse shape represented by a Bessel function \( J_0(\cdot) \), so that its field is concentrated in the area surrounding the propagation axis \( z \). Moreover, Eq. (1.11) tells us that the Bessel beam keeps its transverse shape (which is therefore invariant) while propagating, with central spot \( \Delta \rho = 2.405c/(\omega \sin \theta) \).

The ideal Bessel beam is, however, not a square-integrable function and thus possesses infinite energy (i.e., it cannot be produced experimentally).

But we can have recourse to truncated Bessel beams, generated by finite apertures. In this case the (truncated) Bessel beams are still able to travel a long distance while maintaining their transfer shape, as well as their speed, approximately unchanged [40,69,70]; that is, they still possess a large field depth. For instance, the depth of field of a Bessel beam generated by a circular finite aperture with radius \( R \) is given by

\[
Z_{max} = \frac{R}{\tan \theta}, \tag{1.12}
\]
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FIGURE 1.6 Comparison of a Gaussian (a) and a truncated Bessel beam (b). The Gaussian beam doubles its initial transverse width after 3 cm, and after 6 cm its intensity decays by a factor of 10. By contrast, the Bessel beam keeps its transverse shape approximately up to a distance of 85 cm.

where $\theta$ is the beam axicon angle. In the finite-aperture case, the Bessel beam can no longer be represented by Eq. (1.11). One has to calculate it by the scalar diffraction theory: using, for example, Kirchhoff’s or Rayleigh–Sommerfeld’s diffraction integrals. But until reaching $Z_{\text{max}}$, one may still use Eq. (1.11) for describing it approximately, at least in the surrounding of the axis $\rho = 0$: namely, for $\rho \ll R$.

To realize how much a truncated Bessel beam succeeds in resisting diffraction, let us also consider a Gaussian beam, with the same frequency and central spot, and compare their field depths. In particular, let us assume for both beams $\lambda = 0.63 \mu m$, and an initial central spot size $\Delta \rho_0 = 60 \mu m$. The Bessel beam will possess axicon angle $\theta = \arcsin(2.405c/(\omega \Delta \rho_0)) = 0.004$ rad. Figure 1.6 depicts the behavior of the two beams for a Bessel beam circular aperture with radius 3.5 mm. We can see how the Gaussian beam doubles its initial transverse width after 3 cm, and after 6 cm its intensity becomes an order of magnitude smaller. By contrast, the truncated Bessel beam keeps its transverse shape until distance $Z_{\text{max}} = R / \tan \theta = 85$ cm. Afterward, the Bessel beam decays rapidly as a consequence of the sharp cut performed on its aperture (which is also responsible for the intensity oscillations suffered by the beam along its propagation axis and for the fact that eventually the feeding waves coming from the aperture get faint at a certain point).

The zeroth-order (axially symmetric) Bessel beam is simply one example of a localized beam. Further examples are higher-order (not cylindrically symmetric) Bessel beams

$$\psi(\rho, \phi, z; t) = J_\nu \left( \frac{\omega_0}{c} \sin \theta \rho \right) \exp(i\nu\phi) \exp \left[ i \frac{\omega_0}{c} \cos \theta \left( z - \frac{c}{\cos \theta} t \right) \right],$$

(1.13)

and Mathieu beams [68].
Ordinary X-Shaped Pulse  
Following the procedure adopted earlier, let us construct pulses by using spectral functions of the type 
\[ S(k_\rho, \omega) = \frac{\delta(k_\rho - (\omega/c) \sin \theta)}{k_\rho} F(\omega), \]  
(1.14)

where this time the Dirac delta function furnishes the spectral space–time coupling \( k_\rho = (\omega \sin \theta)/c \). The function \( F(\omega) \) is, of course, the frequency spectrum; for the moment it is left undetermined.

On plugging Eq. (1.14) into Eq. (1.5) we obtain 
\[ \psi(\rho, z, t) = \int_{-\infty}^{\infty} F(\omega) J_0 \left( \frac{\omega}{c} \sin \theta \rho \right) \exp \left[ \frac{\omega}{c} \cos \theta \left( z - \frac{c}{\cos \theta} t \right) \right] d\omega. \]  
(1.15)

It is easy to see that \( \psi \) will be a pulse of the type 
\[ \psi = \psi(\rho, z - Vt) \]  
(1.16)

with a speed \( V = c/\cos \theta \) independent of the frequency spectrum \( F(\omega) \). Such solutions are known as X-shaped pulses and are localized (nondiffractive) waves in the sense that they maintain their spatial shape during propagation (see, e.g., [19,20,42] and references therein).

At this point, some remarkable observations are in order:

1. When a pulse consists of a superposition of waves (in this case, Bessel beams) all endowed with the same phase velocity \( V_{ph} \) (in this case, with the same axicon angle) independent of their frequency, it is known that the phase velocity (in this case \( V_{ph} = c/\cos \theta \)) becomes the group velocity [64,72]; that is, \( V = c/\cos \theta > c \). In this sense, the X-shaped waves are called superluminal localized pulses (cf., e.g., [20] and references therein).

2. Even if their group velocity is superluminal, such pulses do not contradict standard physics, having been found in what precedes on the basis of the wave equations—in particular, Maxwell’s equations [20,35]—only. Indeed, as we shall see more clearly in the Appendix, their existence can be understood within special relativity itself [14,18,20,55–57], on the basis of its ordinary postulates [1]. Actually, let us repeat that such pulses are fed by waves originating at the aperture and carrying energy at the standard speed \( c \) (the velocity of light in the electromagnetic case and the velocity of sound in the particular medium considered in the acoustic [21] case). We can become convinced of the possibility of realizing superluminal X-shaped pulses by imagining the simple ideal case of a negligibly sized superluminal source \( S \) endowed with speed \( V > c \) in vacuum and emitting electromagnetic waves \( W \) (each traveling with the invariant speed \( c \)). The electromagnetic waves will be internally tangent to an enveloping cone \( C \) having \( S \) as its vertex and the propagation line \( z \) of the source as its axis [1,18]: This is completely analogous to what happens for an airplane that
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FIGURE 1.7  The truncated X-waves considered in this chapter, as predicted by SR (all wave equations have an intrinsic relativistic structure), must have a leading cone in addition to the rear cone, such a leading cone having a role for the peak stability [19]: For example, when producing a finite conic wave truncated in both space and time, SR theory suggested, in the simplest case, use of a dynamic antenna emitting radiation cylindrically symmetrical in space and symmetric in time, for a better approximation to what Courant and Hilbert [16] called an undistorted progressing wave (see below).

moves in air with constant supersonic speed. The waves $W$ interfere mainly negatively inside the cone $C$ and constructively on its surface. We can place a plane detector orthogonal to $z$ and record magnitude and direction of the $W$ waves that hit it, as (cylindrically symmetric) functions of position and time. It will be enough, then, to replace the plane detector with a plane antenna that emits (instead of recording) exactly the same (axially symmetric) space–time pattern of waves $W$, for constructing a cone-shaped electromagnetic wave $C$ that will propagate with superluminal speed $V$ (of course, no longer with a source at its vertex), even if each wave $W$ travels with the invariant speed $c$. Once more, this is exactly what would happen in the case of a supersonic airplane (in which case, $c$ is the sound speed in air; for simplicity, assume the observer to be at rest with respect to the air). For further details, see the references cited. Actually, by suitable superpositions and interference of speed-$c$ waves, one can obtain pulses that are more and more localized in the vertex region [42], that is, very localized field “blobs” traveling with superluminal group velocity. This has nothing to do with the illusory “scissors effect,” since such blobs, along their depth of field, are a priori able, for example, to get two successive (weak) detectors, located at distance $L$, clicking after a time smaller than $L/c$. Incidentally, an analysis of the above-mentioned case (that of a supersonic plane or a superluminal charge) led, as expected [1], to the simplest type of X-shaped pulse [18]. It might be useful, finally, to recall that special relativity (SR) (even the wave equations have an internal relativistic structure!) implies also considering the forward cone (see Fig. 1.7). The truncated X-waves considered in this chapter, for instance, must have a leading cone in addition to the rear cone, such a leading cone having a role for the peak stability [19]. For example, in the approximate case in which we produce a finite conic wave truncated in both space and time, SR theory suggested the biconic shape (symmetrical in space with respect to the vertex $S$) to be a better approximation to a rigidly traveling wave (so that SR suggests using a dynamic antenna emitting a radiation cylindrically symmetrical in space and symmetrical in time, for a better approximation to an undistorted progressing wave).
3. Any solutions that depend on $z$ and on $t$ only through the quantity $z - Vt$, such as Eq. (1.15), will appear the same to an observer traveling along $z$ with speed $V$, whatever the value of $V$ (subluminal, or superluminal). That is, such a solution will propagate rigidly with speed $V$ (and in fact, there exist superluminal, luminal, and subluminal localized waves). This explains further why the X-shaped pulses, after having been produced, will travel almost rigidly at speed $V$ (in this case, a faster-than-light group velocity) all along their depth of field. To be even clearer, let us consider a generic function, depending on $z - Vt$ with $V > c$ (incidentally, this wave function is simply the classic X-shaped wave in Cartesian coordinates).

Let us verify that it is a solution to the wave equation

$$
\nabla^2 \Phi(x, y, z, t) - \frac{1}{c^2} \frac{\partial^2 \Phi(x, y, z, t)}{\partial^2 t} = 0.
$$

On setting

$$
R \equiv \sqrt{(b - ic(z - Vt))^2 + (V^2 - c^2)(x^2 + y^2)},
$$

one can write $\Phi = a/R$ and evaluate the second derivatives

$$
\begin{align*}
\frac{1}{a} \frac{\partial^2 \Phi}{\partial^2 z} & = \frac{c^2}{R^3} - \frac{3c^2}{R^5} [b - ic(z - Vt)]^2 \\
\frac{1}{a} \frac{\partial^2 \Phi}{\partial^2 x} & = -\frac{V^2 - c^2}{R^3} + 3 (V^2 - c^2)^2 \frac{x^2}{R^5} \\
\frac{1}{a} \frac{\partial^2 \Phi}{\partial^2 y} & = -\frac{V^2 - c^2}{R^3} + 3 (V^2 - c^2)^2 \frac{y^2}{R^5} \\
\frac{1}{a} \frac{\partial^2 \Phi}{\partial^2 t} & = \frac{c^2 V^2}{R^3} - \frac{3c^2 V^2}{R^5} [b - ic(z - Vt)]^2,
\end{align*}
$$

from which

$$
\begin{align*}
\frac{1}{a} \left( \frac{\partial^2 \Phi}{\partial^2 z} - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial^2 t} \right) & = -\frac{V^2 - c^2}{R^3} + 3 (V^2 - c^2)^2 \frac{[b - ic(z - Vt)]^2}{R^5} \\
\frac{1}{a} \left( \frac{\partial^2 \Phi}{\partial^2 x} + \frac{\partial^2 \Phi}{\partial^2 y} \right) & = -2 \frac{V^2 - c^2}{R^3} + 3 (V^2 - c^2)^2 \frac{x^2 + y^2}{R^5}.
\end{align*}
$$
1.2 MORE DETAILED INFORMATION

From these two equations, based on the previous definition, one finally gets

\[
\frac{1}{a} \left( \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} \right) = 0,
\]

which is simply the (d’Alembert) wave equation (1.18). In conclusion, the function \( \Phi \) is a solution of the wave equation even if it represents a pulse (Selleri says a “signal”) propagating with superluminal speed.

After the three important observations that we made above, let us return to our evaluations with regard to X-type solutions to the wave equations. Let us now consider, for example, the particular frequency spectrum \( F(\omega) \) in Eq. (1.15), given by

\[
F(\omega) = H(\omega) \frac{a}{V} \exp \left( -\frac{a}{V} \omega \right),
\]

(1.20)

where \( H(\omega) \) is the Heaviside step function and \( a \) is a positive constant. Then, Eq. (1.15) yields

\[
\psi(\rho, z - Vt) \equiv X = \frac{a}{\sqrt{(a - i\xi)^2 + [(V^2/c^2) - 1]\rho^2}},
\]

(1.21)

with \( \xi \equiv z - Vt \). This solution is the well-known ordinary, or “classic,” X-wave, which is a simple example of an X-shaped pulse [19,20]. Notice that function (1.20) contains mainly low frequencies, so that the classic X-wave is suitable for low frequencies only. Figure 1.8 depicts the real part of an ordinary X-wave with \( V = 1.1c \) and \( a = 3 \) m.
Solutions (1.15), and in particular the pulse (1.21), have infinite depth of field as well as infinite energy. Therefore, as done in the Bessel beam case, we should go on to truncated pulses originating from a finite aperture. Afterward, our truncated pulses will keep their spatial shape (and their speed) all along the depth of field

\[ Z = \frac{R}{\tan \theta}, \] (1.22)

where, as before, \( R \) is the aperture radius and \( \theta \) is the axicon angle.

**Further Observations**  It is not strictly correct to call localized waves nondiffractive, since diffraction more or less affects all waves obeying Eq. (1.1). However, all localized waves (both beams and pulses) possess the remarkable self-reconstruction property: That is, when diffracting during propagation, localized waves rebuild their shape immediately [71,78,117] (even after obstacles with size much larger than the implied wavelengths, provided, of course, that it is smaller than the aperture size), due to their particular spectral structure, as shown in more detail in other chapters. In particular, ideal localized waves (those with infinite energy and depth of field) are able to rebuild for an infinite time, whereas, as we have seen, finite-energy (truncated) waves can rebuild, and thus resist diffraction effects, only along a certain depth of field.

Let us stress again that interest in localized waves (especially from the point of view of applications) lies in the fact that they are almost nondiffractive, and not in their group velocity. From this point of view, superluminal, luminal, and subluminal localized solutions are equally interesting and suited to important applications.

Actually, localized waves are not all restricted to the X-shaped, superluminal waves corresponding to the integral solution (1.15) of the wave equation; and, as we said earlier, three classes of localized pulses exist: superluminal (with speed \( V > c \)), luminal (\( V = c \)), and subluminal (\( V < c \)), all of them with or without axial symmetry, and all corresponding to a single unified mathematical background. This issue is touched on again in the book. Incidentally, elsewhere we have addressed such topics as (1) the construction of infinite families of generalizations of the classic X-shaped wave, with energy concentrated more and more around the vertex, as shown in Fig. 1.9; (2) the behavior of some finite total-energy superluminal localized solutions (SLS); (3) a way to build up a new series of SLSs to Maxwell’s equations suitable for arbitrary frequencies and bandwidths; and (4) questions related to dispersive media: In Chapter 2 we return to some of these points. We add that X-shaped waves have also been produced easily in nonlinear media [4], as described in Chapter 9. A more technical introduction to the subject of localized waves (particularly with respect to superluminal X-shaped waves) may be found in [55].

**APPENDIX: THEORETICAL AND EXPERIMENTAL HISTORY**

In this mainly historical section, we first describe from a theoretical point of view the most intriguing localized solutions to the wave equation: the superluminal solutions
THEORETICAL AND EXPERIMENTAL HISTORY

FIGURE 1.9 (a) Square magnitude (arbitrary units) of the classic, X-shaped superluminal localized solution (SLS) to the wave equation, with $V = 5c$ and $a = 0.1$ m. Families of infinite SLSs exist, which generalize the classic X-shaped solution; for instance, a family of SLSs obtained [42] by suitably differentiating the classic X-wave. (b) The first SLS (corresponding to the first differentiation) with the same parameters. Successive solutions in such a family are more localized around their vertex. The quantity $\rho$ is the distance in meters from the propagation axis $z$, while the quantity $\zeta$ is the V-cone variable [42] (also in meters) $\zeta \equiv z - Vt$, with $V \geq c$. Since all these solutions depend on $z$ only via the variable $\zeta$, they propagate “rigidly” (i.e., without distortion and thus are called localized, or nondispersive). Here we assume propagation in the vacuum (or in a homogeneous medium).

(SLSs), in particular the X-shaped pulses. To start with, we recall their geometrical interpretation within special relativity (SR). Afterward, to help resolve possible doubts, we present a bird’s-eye view of the various experimental sectors of physics in which superluminal motions seem to appear: in particular, of the experiments with evanescent waves (and/or tunneling photons) or with the SLSs we are more interested in here. In some parts of this appendix the motion line is called $x$ rather than $z$, but that should not present any problems of interpretation.

The subject of superluminal ($V^2 > c^2$) objects or waves has a long history, beginning prior to special relativity in papers by J. J. Thomson and A. Sommerfeld, among others. However, with the development of special relativity, the conviction spread that the speed $c$ of light in a vacuum was the upper limit of speed possible. For example, R. C. Tolman (in 1917) believed that he had shown by his “paradox” that the existence of particles endowed with speeds higher than $c$ would allow sending information into the past. The problem was not tackled again until the 1950s and 1960s, in particular after the work by Bilaniuk et al. [89] and later by one of the present authors with Mignani et al. [80,81], as well as (confining ourselves at present to theoretical research) by H. C. Corben and others. The first experimental attempts were performed by T. Alväger et al.

We wish to face the still unusual issue of the possible existence of superluminal wavelets and objects within standard physics and SR, since at least four different experimental sectors of physics seem to support such a possibility (apparently confirming some long-standing theoretical predictions [1,14,81,104]). The experimental
review will be necessarily short, but we provide the reader with further bibliographical information (limited for the sake of brevity to the twentieth century).

**Historical Recollections: Theory**

Long ago a simple theoretical framework based on the space–time geometrical methods of SR was proposed [1,80] which appears to incorporate superluminal waves and objects and to predict [14] superluminal X-shaped waves without violating the principles of relativity. A suitable choice of the postulates of SR (equivalent, of course, to other, more common choices) consists in the standard principle of relativity and space–time homogeneity and space isotropy. It follows that one and only one invariant speed exists; and experience shows invariant speed to be the speed of light, $c$, in vacuum: The essential role of $c$ in SR is due simply to its invariance, not to the fact that it is a maximal or minimal speed. No sub- or superluminal objects or pulses can be endowed with an invariant speed, so in SR their speed cannot play the same essential role played by the light-speed $c$. Indeed, $c$ turns out to be a limiting speed, but any limit possesses two sides and can be approached a priori from both below and above (see Fig. 1A.1). As Sudarshan put it, from the fact that no one can climb over the Himalayas, people in India should not conclude that there are no people north of the Himalayas; actually, speed-$c$ photons exist, which are born, live, and die “at the top of the mountain,” without any need to performing the impossible task of accelerating from rest to the speed of light. (Actually, the ordinary formulation of SR is too restricted: Even leaving superluminal speeds aside, it can easily be broadened to include antimatter [1,57,58].)

An immediate consequence is that the quadratic form $c^2 dt^2 - dx^2 = dx_{\mu} dx^{\mu}$, called $ds^2$, with $dx^2 = dx^2 + dy^2 + dz^2$, is invariant except for its sign. The quantity $ds^2$ is a four-dimensional length-element square along the space–time path of any object. Corresponding to a positive (negative) sign, we have subluminal (superluminal) *Lorentz transformations* [LTs]. Ordinary subluminal LTs are known to leave exactly the quadratic forms $dx_{\mu} dx^{\mu}$, $dp_{\mu} dp^{\mu}$, and $dx_{\mu} dp^{\mu}$ invariant (where the $p_{\mu}$ are components of the energy-impulse four-vector), whereas superluminal LTs have to change only the sign of such quadratic forms. This is enough to deduce some important consequences, such as the fact that a superluminal charge has to behave as a magnetic monopole (in the sense specified in [1] and references therein).
A more important consequence for us is that the simplest subluminal object (Fig. 1A.2), a spherical particle at rest (which appears ellipsoidal due to Lorentz contraction, at subluminal speeds $v$) will appear [1,14,20] to occupy the cylindrically symmetrical region bounded by a two-sheeted rotation hyperboloid and an indefinite double cone (Fig. 1A.2d) for superluminal speeds $V$. In the figure the motion is along the $x$-axis. In the limiting case of a pointlike particle, we obtain only a double cone. Such a result is reached simply by writing down the equation of the world-tube of a subluminal particle and transforming it merely by changing the sign of the quadratic forms entering that equation. Thus, in 1980–1982, it was predicted [14] that the simplest superluminal object appears not as a particle, but as a field or, rather, as a wave; namely, as an X-shaped pulse, the cone semiangle $\alpha$ being given (with $c = 1$) by $\cot \alpha = \sqrt{V^2 - 1}$. Such X-shaped pulses will move rigidly with speed $V$ along their motion direction. In fact, any X-pulse can be regarded at each instant of time as the superluminal Lorentz transform of a spherical object, which of course moves without any deformation in vacuum, or in a homogeneous medium, as time elapses. A three-dimensional picture of Fig. 1A.2d appears in Fig. 1A.3, where its annular intersections with a transverse plane are shown (see [14]). The X-shaped waves considered here are merely the simplest ones: If one started not from an intrinsically spherical or pointlike object, but from a nonspherically symmetric particle, a pulsating (contracting and dilating) sphere, or a particle oscillating back and forth along the direction of motion, their superluminal Lorentz transforms would be more and more complicated. The X-waves above are typical for a superluminal object, however, as the spherical or pointlike shape are typical for a subluminal object.

Incidentally, it has been believed for a long time that superluminal objects would have allowed sending information into the past, but such problems with causality seem to be solvable within SR. Apparently, once SR is generalized to include superluminal
FIGURE 1A.3 Intersections of the superluminal object $T$ in Fig. 1A.2d with planes $P$ orthogonal to its line of motion (the $x$-axis). For simplicity, we again assumed the object to be spherical in its rest frame and the cone vertex $C$ to coincide with the origin $O$ for $t = 0$. Such intersections evolve in time so that the same pattern appears on a second plane, shifted by $\Delta x$, after the time $\Delta t = \Delta x / V$. On each plane, as time elapses, the intersection is therefore predicted by extended SR to be a circular ring which, for negative times, goes on shrinking until it is reduced to a circle and then to a point (for $t = 0$); afterward, such a point again becomes a circle and then a circular ring that goes on broadening $[1, 14, 20]$. [Notice that if the object is not spherical when at rest (but, e.g., is ellipsoidal in its own rest frame), the axis of $T$ will no longer coincide with $x$, but its direction will depend on the speed $V$ of the tachyon itself.] For the case in which the space extension of the superluminal object $T$ is finite, see $[14]$.

(From $[1, 14]$.)

objects or pulses, no signal traveling backward in time is left. For a solution of those causal paradoxes, see $[58, 104]$ and references therein.

To address the problem, even within this elementary context, of the production of an $X$-shaped pulse such as the one depicted in Fig. 1A.3 (perhaps truncated in space and time by use of a finite antenna radiating for a finite time), all the considerations described under observation 2 of the section “Ordinary X-Shaped Pulse” (Section 1.2.1) come into order: And, here, we simply refer to them. Those considerations, together with the present ones (related, e.g., to Fig. 1A.3), suggest that the simplest antenna consists of a series of concentric annular slits or transducers (as in Fig. 1.2), which suitably radiate following specific time patterns (see, e.g., $[102]$ and references therein). Incidentally, the process above can lead to a very simple type of $X$-shaped wave.

From the present point of view, it is rather interesting to note that, during the last 15 years, $X$-shaped waves have actually been found as solutions to Maxwell and wave equations (recall once more that the form of any wave equations is intrinsically relativistic). To see more deeply the connection existing between what predicted by SR (see, e.g., Figs. 1A.2 and 1A.3) and the localized $X$-waves (mathematically and
experimentally constructed in recent times), below we look in detail at the problem of the X-shaped field created by a superluminal electric charge, by following a recent paper [18].

**X-Shaped Field Associated with a Superluminal Charge**

It is well known by now that Maxwell’s equations admit of wavelet-type solutions endowed with arbitrary group velocities \(0 < v_g < \infty\). We again confine ourselves, as above, to localized solutions moving rigidly, in particular to superluminal solutions (SLSs), the most interesting of which turned out to be X-shaped. As we already know, SLSs have been produced in a number of experiments, always by suitable interference of ordinary-speed waves. Here we show, by contrast, that even a superluminal charge creates an electromagnetic X-shaped wave, in agreement with what has been predicted within SR [1, 14]: namely, that, on the basis of Maxwell equations, one is able to evaluate the field associated with a superluminal charge (at least under the rough approximation of pointlikeness). As noted earlier, it will result to a very simple example of true X-wave.

Indeed, when based on the ordinary postulates but not restricted to subluminal waves and objects (i.e., in its extended version), SR theory predicted the simplest X-shaped wave to be the wave corresponding to the electromagnetic field created by a superluminal charge [18, 79]. It seems really important to evaluate such a field, at least approximately, by following [18].

**Toy Model of a Pointlike Superluminal Charge**

We begin by considering, formally, a pointlike superluminal charge, even if the hypothesis of pointlikeness (already unacceptable in the subluminal case) is totally inadequate in the superluminal case [1]. Then we consider the ordinary vector potential \(A^\mu\) and a current density \(j^\mu \equiv (0, 0, j_z; j^0)\) flowing in the \(z\)-direction (notice that the motion line here is the \(z\)-axis). On assuming the fields to be generated by the sources only, we have that \(A^\mu \equiv (0, 0, A_z; \phi)\), which, when adopting the Lorentz gauge, obeys the equation \(A^\mu = j^\mu\). We can write such a nonhomogeneous wave equation in cylindrical coordinates \((\rho, \theta, z; t)\); for axial symmetry [which requires a priori that \(A^\mu = A^\mu(\rho, z; t)\)], when choosing the \(V\)-cone variables \(\zeta \equiv z - Vt, \eta \equiv z + Vt\), with \(V^2 > c^2\), we arrive [18] at the equation

\[
\left[ -\rho \frac{\partial}{\partial \rho} \left( \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \right) + \frac{1}{\gamma^2} \frac{\partial^2}{\partial \zeta^2} + \frac{1}{\gamma^2} \frac{\partial^2}{\partial \eta^2} - 4 \frac{\partial^2}{\partial \zeta \partial \eta} \right] A^\mu(\rho, \zeta, \eta) = j^\mu(\rho, \zeta, \eta),
\]

(1A.1)

where \(\mu\) assumes the two values \(\mu = 3, 0\) only, so that \(A^\mu \equiv (0, 0, A_z; \phi)\) and \(\gamma^2 \equiv (V^2 - 1)^{-1}\). (Notice that, whenever convenient, we set \(c = 1\).) Let us now suppose \(A^\mu\) to be actually independent of \(\eta\); namely, \(A^\mu = A^\mu(\rho, \zeta)\). Due to Eq. (1A.1), we also have \(j^\mu = j^\mu(\rho, \zeta)\), and therefore, \(j_z = Vj^0\) (from the continuity equation) and \(A_z = V\phi/c\) (from the Lorentz gauge). Then, by setting \(\psi \equiv A_z\), we end with two
LOCALIZED WAVES: A HISTORICAL AND SCIENTIFIC INTRODUCTION

FIGURE 1A.4 Behavior of the field $\psi \equiv A_z$ generated by a charge supposed to be superluminal, as a function of $\rho$ and $\zeta \equiv z - Vt$, evaluated for $\gamma = 1$ (i.e., for $V = c\sqrt{2}$), according to [18]. (Of course, we skipped the points in which $\psi$ must diverge: namely, the vertex and the cone surface.)

equations which allow us to analyze the possibility and consequences of having a superluminal pointlike charge, $e$, traveling with constant speed $V$ along the $z$-axis ($\rho = 0$) in the positive direction, in which case $j_z = eV\delta(\rho)/\rho\delta(\zeta)$. Indeed, one of those two equations becomes the hyperbolic equation

$$\left[-\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\gamma^2} \frac{\partial^2}{\partial \zeta^2}\right] \psi = eV \frac{\delta(\rho)}{\rho} \delta(\zeta). \quad (1A.2)$$

which can be solved [18] in a few steps: first, by applying (with respect to the variable $\rho$) the Fourier–Bessel (FB) transformation $f(x) = \int_{0}^{\infty} \Omega f(\Omega) J_0(\Omega x) d\Omega$, the quantity $J_0(\Omega x)$ being the ordinary zero-order Bessel function; second, by applying the ordinary Fourier transformation with respect to the variable $\zeta$ (passing from $\zeta$ to the variable $\omega$); and third, by performing the corresponding inverse Fourier and FB transformations. Afterward it is enough to have recourse to formulas (3.723.9) and (6.671.7) of [82], still with $\zeta \equiv z - Vt$, to enable writing the solution of Eq. (1A.2) in the form

$$\psi(\rho, \zeta) = \begin{cases} 
0 & \text{for } 0 < \gamma |\zeta| < \rho \\
\frac{V}{\sqrt{\zeta^2 - \rho^2(V^2 - 1)}} & \text{for } 0 \leq \rho < \gamma |\zeta|. 
\end{cases} \quad (1A.3)$$

In Fig. 1A.4 we show our solution, $A_z \equiv \psi$, as a function of $\rho$ and $\zeta$, evaluated for $\gamma = 1$ (i.e., for $V = c\sqrt{2}$). Of course, we skipped the points in which $A_z$ must diverge: namely, the vertex and the cone surface.
The spherical equipotential surfaces of an electrostatic field created by a charge at rest get transformed into two-sheeted rotation hyperboloids contained inside an unlimited double cone when the charge travels at superluminal speed (see [1,18]). This figures shows that a superluminal charge traveling at constant speed in a homogeneous medium such as a vacuum does not lose energy [79]. Note that this double cone has nothing to do with the Cherenkov cone. (From [1].)

For comparison, one may recall that the classic X-shaped solution [19] of the homogeneous wave equation (shown, e.g., in Figs. 1.8, 1.9, and 1A.3) has the form (with \(a > 0\))

\[
X = \frac{V}{\sqrt{(a - i \xi)^2 + \rho^2(V^2 - 1)\}}.
\]

The second of Eqs. (1A.3) includes the expression (1A.4), given by the spectral parameter \([42,63]\) \(a = 0\), which indeed corresponds to the nonhomogeneous case (the fact that for \(a = 0\) these equations differ for an imaginary unit will be discussed elsewhere).

It is rather important at this point to note that such a solution, Eq. (1A.3), does represent a wave existing only inside the (unlimited) double cone \(C\) generated by the rotation around the \(z\)-axis of the straight lines \(\rho = \pm \gamma \xi\): This, too, is in full agreement with the predictions of extended SR theory. For an explicit evaluation of the electromagnetic fields generated by the superluminal charge (and of their boundary values and conditions), we confine ourselves here to quoting [18]. Incidentally, the same results found by following the procedure described above can be obtained by starting from the four-potential associated with a subluminal charge (e.g., an electric charge at rest) and applying to it the suitable superluminal Lorentz "transformation". One should also notice that this double cone does not have much to do with the Cherenkov cone [1,79]; and a superluminal charge traveling at constant speed, in the vacuum, does not lose energy (see, e.g., Fig. 1A.5).
Outside cone \( C \) (i.e., for \( 0 < \gamma |\xi| < \rho \)) we get, as expected, no field, so that one meets a field discontinuity when crossing the double-cone surface. Nevertheless, the boundary conditions imposed by Maxwell’s equations are satisfied by our solution (1A.3), since at each point of the cone surface the electric and magnetic fields are both tangent to the cone (for a discussion of this point, see [18]).

Here, let us stress that, when \( V \to \infty, \gamma \to 0 \), the electric field tends to vanish while the magnetic field tends to the value \( H_\phi = -\pi e/\rho^2 \). This does agree with what we expect from extended SR, which predicts that superluminal charges behave (in a sense) as magnetic monopoles. We refer interested readers to [1,2,80,81], and references therein.

**A Glance at the Experimental State of the Art**

Extended relativity can also provide a better understanding of many aspects of ordinary physics [1], even if superluminal objects (tachyons) did not exist in our cosmos as asymptotically free objects. In any case, at least three or four different experimental sectors of physics seem to suggest the possible existence of faster-than-light motion, or at least of superluminal group velocities. Next, we provide some experimental results obtained in two sectors and mention two others.

**Neutrinos** A long series of experiments begun in 1971 seem to show that the square \( m_0^2 \) of the mass \( m_0 \) of muon-neutrinos, and more recently of electron-neutrinos, is negative; which, if confirmed, would mean that such neutrinos possess an “imaginary mass” and are therefore tachyonic or mainly tachyonic [1,83,84]. (Actually, in extended SR, the dispersion relation for a free superluminal object becomes \( \omega^2 - k^2 = -\Omega^2 \), or \( E^2 - p^2 = -m_0^2 \), and there is no need at all, therefore, for imaginary masses.)

**Galactic Microquasars** As for the apparent superluminal expansions observed in the cores of quasars [85] and, recently, in galactic microquasars [86], they are outside the range of this chapter. Moreover, we note that there exist orthodox interpretations for such astronomical observations, based on [87], that are accepted by the majority of astrophysicists (for a theoretical discussion, see [88]). Here, we mention only that simple geometrical considerations in Minkowski space show that a single superluminal source of light would appear [1,88]: (1) initially, in the “optical boom” phase (analogous to the acoustic boom produced by an airplane traveling at a constant supersonic speed), as an intense source that suddenly comes into view; and (2) which afterward seems to split into two objects receding from one another with speed \( V > 2c \) (all of this being similar to what is actually observed according to [86]).

**Evanescent Waves and Tunneling Photons** Within quantum mechanics (specifically, in the tunneling processes), it has been shown that tunneling time (first evaluated as a simple Wigner’s “phase time” and later calculated through the analysis of wave
packet behavior) does not depend [90] on barrier width in the case of opaque barriers (the Hartman effect). This implies superluminal and arbitrarily large group velocities \( V \) inside sufficiently long barriers (see Fig. 1A.6). Experiments that might verify this prediction using, say, electrons or neutrons are difficult and rare [65,91]. Luckily enough, however, the Schrödinger equation in the presence of a potential barrier is mathematically identical to the Helmholtz equation for an electromagnetic wave propagating, for instance, down a metallic waveguide (along the \( x \)-axis): as shown, for example, in [118]; and a barrier height \( U \) greater than the electron energy \( E \) corresponds (for a given wave frequency) to a waveguide of transverse size lower than the cutoff value. In the context of this example, a segment of “undersized” guide therefore behaves as a barrier for the wave (photonic barrier) as well as any other photonic bandgap filters. Thus, like a particle inside a quantum barrier, the wave therein assumes an imaginary momentum or wave number and so is damped exponentially along \( x \) (see, e.g., Fig. 1A.7). It becomes an evanescent wave (returning to normal propagation, even if with reduced amplitude, when the narrowing ends and the guide returns to its initial transverse size). Thus, a tunneling experiment can be simulated by having recourse to evanescent waves (for which the concept of group velocity can be properly extended: see the first of [57]).

The fact that evanescent waves can travel with superluminal speeds (see, e.g., Fig. 1A.8) has actually been verified in a series of famous experiments. Work performed from 1992 onward by Nimtz et al. in Cologne [106], Steinberg et al. at Berkeley
FIGURE 1A.7  The damping that takes place inside a barrier reduces the amplitude of a tunneling wave packet, imposing a practical limit on the barrier length. (From [57].)

[105], Mugnai et al. in Florence [23], and by others in Vienna, Orsay, Rennes, and other locations [95,107] verified that tunneling photons travel with superluminal group velocities. Such experiments even raised a great deal of interest [93,96,108] within the nonspecialized press and were reported in Scientific American, Nature, New Scientist, and other journals. We should add that, since also extended SR had predicted [109]

![Diagram](attachment:figure1a7.png)

**FIGURE 1A.7** Simulation of tunneling by experiments with evanescent classical waves (see the text), which were also predicted to be superluminal on the basis of extended SR [109]. The figure shows one of the measurement results by Nimtz et al. [94]: the average beam speed while crossing the evanescent region (i.e., segment of undersized waveguide, or “barrier”) as a function of its length. As predicted theoretically [90,109], such an average speed exceeds c for sufficiently long barriers. Further results appeared in [99] and are reported below (see Figs. 1A.11 and 1A.12).
evanescent waves to be endowed with faster-than-c speeds, the entire matter therefore appears to be theoretically self-consistent. The debate in the current literature does not refer to the experimental results (which can be reproduced correctly even by numerical simulations [73,74] based on Maxwell’s equations only; see Figs.1A.9 and 1A.10), but rather, to the question of whether they do or do not allow signals or information to be sent with superluminal speed (see, e.g., [66]).

In the experiments mentioned above, there is substantial attenuation of the pulses (see Fig. 1A.7) during tunneling (or during propagation in an absorbing medium). However, by employing “gain doublets,” undistorted pulses have been observed to propagate at superluminal group velocity with only a small change in amplitude (see, e.g., [97]).

Let us emphasize that some of the most interesting experiments in this series seem to be those with two or more barriers (e.g., with two gratings in an optical fiber or with two segments of undersized waveguide separated by a piece of normal-sized waveguide; Fig. 1A.11). For suitable frequency bands (i.e., for tunneling far from resonances) it was found that the total crossing time does not depend on the length of the intermediate (normal) guide, that is, that the beam speed along it is infinite [91,100,111]. This does agree with what we predicted within quantum mechanics for nonresonant tunneling through two successive opaque barriers [100] (Fig. 1A.12).
FIGURE 1A.10  Envelope of the signal in (Fig. 1A.9) after traveling a distance $L = 32.96$ mm through an undersized waveguide. Inset (a) shows the initial part (in time) of such arriving signal, and inset (b) shows the peak of the Gaussian pulse that had initially been modulated by centering it at $t = 100$ ns. One can see that its propagation took zero time, so that the signal traveled with infinite speed. The numerical simulation is based on Maxwell’s equations only. Going from Fig. 1A.9 to Fig. 1A.10, one verifies that the signal amplitude is reduced greatly. However, the width of each peak did not change (and this might have some relevance when dealing with a Morse alphabet transmission; see the text).

Such a prediction has been verified first, theoretically, by Aharanov et al. [100], and then experimentally by taking advantage of the circumstance [111] that evanescence regions can consist of a variety of photonic bandgap materials or gratings (from multilayer dielectric mirrors, or semiconductors, to photonic crystals). Indeed, the best experimental confirmation has come by having recourse to two gratings in an optical fiber [99]; see Figs. 1A.13 and 1A.14, in particular the rather peculiar (and quite interesting) results represented by the latter.

FIGURE 1A.11  Very interesting experiments have been performed with two successive barriers (i.e., with two evanescence regions): for example, with two gratings in an optical fiber. This figure refers to the interesting experiment [111] performed with microwaves traveling along a metallic waveguide, the waveguide being endowed with two classical barriers (undersized guide segments). See the text. (From [57].)
FIGURE 1A.12 Scheme of a nonresonant tunneling process through two successive (opaque) quantum barriers. Far from resonances, the (total) phase time for tunneling through the two potential barriers depends on neither the barrier widths nor the distance between the barriers (this is “the generalized Hartman effect”) [91,98,100]. See the text.

FIGURE 1A.13 Realization of the quantum-theoretical setup represented in Fig. 1A.12 using as classical (photonic) barriers two gratings in an optical fiber [98]. The corresponding experiment has been performed by Longhi et al. [99].

FIGURE 1A.14 Off-resonance tunneling time versus barrier separation for the rectangular symmetric double-barrier frequency bandgap (FBG) structure considered in [99] (see Fig. 1A.13). The solid line is the theoretical prediction based on group delay calculations; the dots are the experimental points obtained by time delay measurements (the dashed curve is the transit time expected from the input to the output planes for a pulse tuned far away from the stopband of the FBGs). The experimental results [99], as well as the early results in [111], confirm the theoretical prediction of a generalized Hartman effect: in particular, the independence of the total tunneling time from the distance between the two barriers.
We may also note another topic that is arousing more and more interest [97]. Even if all the ordinary causal paradoxes seem to be solvable [1,57,58], one must also bear in mind that, whenever an object $\mathcal{O}$ traveling with superluminal speed is encountered, we may have to deal with negative contributions to the tunneling times [1,91,112], and this should not be regarded as nonphysical. In fact, whenever an object $\mathcal{O}$ (e.g., particle, electromagnetic pulse) overcomes [1,58] the infinite speed with respect to a certain observer, it will appear later to the same observer as the anti-object $\overline{\mathcal{O}}$ traveling in the opposite direction in space [1,58,104]. For instance, when going from the lab frame to a frame $\mathcal{F}$ moving in the same direction as the particles or waves entering the barrier region, the object $\mathcal{O}$ penetrating the final part of the barrier (with almost infinite speed [73,90–92], as in Figs. 1A.6) will appear in frame $\mathcal{F}$ as an anti-object $\overline{\mathcal{O}}$ crossing that portion of the barrier in the opposite space direction [1,58,104]. In the new frame $\mathcal{F}$, therefore, such an anti-object $\overline{\mathcal{O}}$ would make a negative contribution to the tunneling time, which could even result in a negative total tunneling time. For clarification, see the references that we have cited. Let us stress here that even the appearance of negative tunneling times has been predicted within extended SR, on the basis of its ordinary postulates, and has been confirmed recently by quantum-theoretical evaluations [3,91]. (In the case of a nonpolarized beam, the wave anti-packet coincides with the initial wave packet; if, however, a photon is endowed with helicity $\lambda = +1$, the anti-photon will bear the opposite helicity, $\lambda = -1$.) For the theoretical point of view, see the papers cited above (in particular, [90,91], and, more specifically, [113]). On the very interesting experimental side, see the intriguing papers cited in [101,114].

Let us add here that it is possible to obtain, via quantum interference effects, dielectrics with refraction indices varying very rapidly as a function of frequency, and also three-level atomic systems, with almost complete absence of light absorption (i.e., with quantum-induced transparency) [115]. The group velocity of a light pulse propagating in such a medium can decrease to very low values, either positive or negative, with no pulse distortion. It is known that experiments have been performed both in atomic samples at room temperature and in Bose–Einstein condensates, which showed the possibility of reducing the speed of light to a few meters per second. Similar, but negative group velocities, interpreted as implying propagation with superluminal speeds thousands of times higher than those that had been considered previously, have also been predicted in the presence of such an electromagnetically induced transparency for light moving in a rubidium condensate [116]. Finally, let us recall that faster-than-$c$ propagation of light pulses can also be (and was, in some cases) observed by taking advantage of the anomalous dispersion near an absorbing line, or nonlinear and linear gain lines (as already seen), or nondispersive dielectric media, or inverted two-level media, as well as of some parametric processes in nonlinear optics (see, e.g., the work of Kurizki et al.).

**Superluminal Localized Solutions to Wave Equations: X-Shaped Waves** This fourth sector is no less important. It came into fashion again when in a series of remarkable works it was rediscovered that any wave equation (e.g., in the electromagnetic
FIGURE 1A.15 The wave equations possess pulse-type solutions that in the subluminal case are ball-like, in agreement with Fig. 1A.2. For additional comments, see the text.

As we know, a remarkable feature of some of these new solutions (which attracted much attention for their possible applications) is that they propagate as localized, nondispersive pulses even because of their self-reconstruction property. It is easy to realize the practical importance, for instance, of a radio transmission carried out by localized beams (independently of their speed); but nondispersive wave packets can be of use even in theoretical physics for a reasonable representation of elementary particles; and so on. Incidentally, from the point of view of elementary particles, the fact that wave equations possess pulse-type solutions that are ball-like in the subluminal case (see Fig 1A.15), can be a source of meditation, as this can have a bearing on the corpuscle–wave duality problem met in quantum physics (besides agreeing, for example, with Fig. 1A.2).

At the cost of repeating ourselves, let us reemphasize that within extended SR it had been found that, whereas the simplest subluminal object conceivable is a small sphere (or a point in the limiting case), the simplest superluminal object (see [14] and Figs. 1A.2 and 1A.3) is an X-shaped wave (or a double cone as its limit), which travels in a homogeneous medium without deforming (i.e., rigidly). It is not without meaning that the most interesting localized solutions to the wave equations happen to be the superluminal solutions, and with the predicted shape. Even more, since from Maxwell’s equations under simple hypotheses one goes on to the usual scalar wave equation for each electric or magnetic field component, one expects the same solutions to exist also in the sectors of acoustic waves, seismic waves, and gravitational
waves; and this has already been demonstrated in the literature for the acoustic case. Actually, such pulses (as suitable superpositions of Bessel beams) were constructed mathematically for the first time by Lu et al. in the field of acoustics and were then called X-waves or X-shaped waves.

It is important for us that X-shaped waves have been indeed produced in experiments with both acoustic and electromagnetic waves; that is, X-pulses were produced that traveled undistorted in their medium with a speed greater than the speed of sound, in the first case, and than the speed of light, in the second case. The first experiment in acoustics was performed by Lu et al. in 1992 at the Mayo Clinic (and their papers received the first IEEE award). In the electromagnetic case, which is certainly more intriguing, superluminal localized X-shaped solutions were first constructed mathematically (cf., e.g., Fig. 1A.16) in [20], and later produced experimentally by Saari and Reivelt [22] using visible light (Fig. 1A.17), and more recently by Mugnai et al. using microwaves [23]. In the theoretical sector the activity has not been less intense: with the goal to build up, for example, analogous new solutions with finite total energy or more suitable for high frequencies, on the one hand, and localized solutions superluminally propagating even along a normal waveguide (see Fig. 1A.18), on the other hand.

Let us recall the problem of producing an X-shaped superluminal wave like the one in Fig. 1A.13, but truncated, of course, in space and time (using a finite antenna
FIGURE 1A.17 Scheme of the experiment by Saari and Reivelt [22], who announced the production in optics of the beams depicted in Fig. 1A.16. In the present figure one can see what was shown by the experimental results: namely, that the X-shaped waves are superluminal. Indeed, running after the plane waves (the latter regularly traveling with speed $c$), they do catch up with the plane waves. Later, an analogous experiment was performed with microwaves by Mugnai et al. [23].

Radiating for a finite time. In such a situation, the pulse is known to keep its localization and superluminality only up to a certain depth of field [i.e., as long as it is fed by the waves arriving (with speed $c$) from the antenna], decaying abruptly afterward [20,40,42]. Various authors, taking account of the time needed to foster such superluminal waves, have concluded that these localized superluminal pulses are unable to transmit information faster than $c$. Many such questions have been discussed in what precedes; for further details, see the second of [20].

In any case, the existence of X-shaped superluminal (or supersonic) pulses seems to constitute, together with, for example, the superluminality of evanescent waves,

FIGURE 1A.18 Elements of one of the trains of X-shaped pulses constructed mathematically in [67], which propagate down a coaxial guide (in the transverse magnetic case): Analogous X-pulses exist (with infinite or finite total energy) for propagation along a normal-sized cylindrical metallic waveguide. [From (67).]
a confirmation of extended SR, a theory [1] based on the ordinary postulates of SR that consequently does not appear to violate any of the fundamental principles of physics. It is curious, moreover, that one of the first applications of X-waves (which takes advantage of their propagation without deformation) has been accomplished in the field of medicine, specifically in ultrasound scanners [24,25], whereas the most important applications of (subluminal!) frozen waves will probably again affect human health problems (e.g., the cancer).

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